Appendix: A counterexample to the conjecture.

Let $n \geq 2$ be an integer. Consider the following system of equations in non-negative integers:

(E_1)	$x_2 = x_1 \cdot x_1$
(E_2)	$x_3 = x_2 \cdot x_2$
$\dots$	$\dots$
(E_{n-1})	$x_n = x_{n-1} \cdot x_{n-1}$
(E_n)	$x_{n+1} = x_n \cdot x_n$
(E_{n+1})	$x_{n+2} = 1$
(E_{n+2})	$x_{n+3} = x_{n+2} + x_{n+2}$
(E_{n+3})	$x_{n+4} = x_{n+3} + x_{n+3}$
(E_{n+4})	$x_{n+6} = x_{n+4} + x_{n+5}$
(E_{n+5})	$x_1 = x_{n+6} + x_{n+2}$
(E_{n+6})	$x_{n+7} = x_{n+6} \cdot x_{n+6}$
(E_{n+7})	$x_{n+9} = x_{n+8} + x_{n+8}$
(E_{n+8})	$x_{n+10} = x_{n+9} + x_{n+2}$
(E_{n+9})	$x_{n+11} = x_{n+7} \cdot x_{n+10}$
(E_{n+10})	$x_{n+1} = x_{n+11} + x_{n+2}$.

Conditions (E_1)–(E_n) guarantee that

(1) $\quad x_{n+1} = x_1^{2^n}$

while (E_{n+1})–(E_{n+4}) imply that

(2) $\quad x_{n+2} = 1$

(3) $\quad x_{n+3} = 2$

(4) $\quad x_{n+4} = 4$

(5) $\quad x_{n+6} \geq 4.$

By (E_{n+1}), (E_{n+7}), (E_{n+8}), we have that

(6) $\quad x_{n+10} = 2x_{n+8} + 1.$

It follows from (E_{n+1}) (E_{n+6}), (E_{n+9}), (E_{n+10}) and (6) that

$$(7) \quad x_{n+1} = x_{n+6}^2(2x_{n+8}+1)+1.$$

Finally, (E_{n+1}), (E_{n+5}) and (1) imply that

$$\begin{array}{lrcl} (8) & x_1 & = & x_{n+6}+1, \\ (9) & x_{n+1} & = & (x_{n+6}+1)^{2^n}, \end{array}$$

which together with (7) gives

$$(10) \quad (x_{n+6}+1)^{2^n} = x_{n+6}^2(2x_{n+8}+1)+1.$$

The left-hand side of (10) can be expanded as

$$1 + 2^n x_{n+6} + \sum_{k=2}^{2^n} \binom{2^n}{k} x_{n+6}^k.$$

Therefore, since $x_{n+6} > 0$ by (5), Eq. (10) implies that

$$(11) \quad 2^n + \sum_{k=2}^{2^n} \binom{2^n}{k} x_{n+6}^{k-1} = x_{n+6}(2x_{n+8}+1).$$

In particular, x_{n+6} divides 2^n. Combining this with (5) we can write

$$(12) \quad x_{n+6} = 2^m \text{ for some integer } m,\ 2 \le m \le n.$$

Lemma 1. *For any positive integer n, for any integer $m \ge 2$ and for any k satisfying $2 \le k \le 2^n$, the number $\binom{2^n}{k}2^{m(k-1)}$ is divisible by 2^{n+1}.*

Proof. Let $v_2(z)$ denote the 2-adic valuation of z. For any k, $2 \le k \le 2^n$, we have

$$\binom{2^n}{k} = \frac{2^n}{k}\binom{2^n-1}{k-1},$$

hence,

$$\begin{aligned} v_2\left(\binom{2^n}{k}2^{m(k-1)}\right) &= v_2\left(\frac{2^n}{k}\binom{2^n-1}{k-1}2^{m(k-1)}\right) \\ &\ge v_2(2^n) - v_2(k) + v_2(2^{m(k-1)}) \\ &= n - v_2(k) + (k-1)m \\ &\ge n - \log_2(k) + 2(k-1). \end{aligned}$$

By an easy induction we obtain that $4^k \ge 8k$ for any $k \ge 2$, which is equivalent to $2(k-1) - \log_2(k) \ge 1$. Therefore,

$$v_2\left(\binom{2^n}{k}2^{m(k-1)}\right) \ge n+1 \text{ for any } k \ge 2.$$

□

Lemma 1 together with (12) imply that $v_2\left(\binom{2^n}{k}x_{n+6}^{k-1}\right) \geq n+1$ for any k satisfying $2 \leq k \leq 2^n$. Consequently, (11) can be rewritten as

(13) $2^n(1+2b) = x_{n+6}(2x_{n+8}+1)$,

where

(14) $b = \sum_{k=2}^{2^n} 2^{-n-1}\binom{2^n}{k}x_{n+6}^{k-1}$

is an integer. Since x_{n+6} is a power of 2 by (12) we conclude that

$$\begin{aligned} (15)\quad x_{n+6} &= 2^n, \\ (16)\quad x_{n+8} &= b = \sum_{k=2}^{2^n} 2^{-n-1}\binom{2^n}{k}2^{n(k-1)} = \sum_{k=2}^{2^n} 2^{-1}\binom{2^n}{k}2^{n(k-2)}. \end{aligned}$$

Eq. (8) gives

(17) $x_1 = 2^n + 1$.

Eqs. (4), (15) and (E_{n+4}) imply that

(18) $x_{n+5} = 2^n - 4$.

In particular, $x_{n+5} \geq 0$ since we assumed $n \geq 2$. Eqs. (E_{n+6}) and (15) give

(19) $x_{n+7} = 2^{2n}$.

Eqs. (E_{n+7}), (E_{n+8}), (E_{n+1}) and (16) imply

$$\begin{aligned} (20)\quad x_{n+9} &= 2b = \sum_{k=2}^{2^n}\binom{2^n}{k}2^{n(k-2)} \\ (21)\quad x_{n+10} &= 1 + 2b = 1 + \sum_{k=2}^{2^n}\binom{2^n}{k}2^{n(k-2)}. \end{aligned}$$

Eqs. (E_{n+9}), (19) and (21) imply

(22) $x_{n+11} = 2^{2n} + \sum_{k=2}^{2^n}\binom{2^n}{k}2^{nk} = \sum_{k=1}^{2^n}\binom{2^n}{k}2^{nk} = (2^n+1)^{2^n} - 1.$

Finally, using (17) and applying (E_1)–(E_n) consecutively, we conclude that

$$(23)\quad \begin{aligned} x_2 &= (2^n+1)^2 \\ x_3 &= (2^n+1)^4 \\ \ldots & \quad \ldots \\ x_n &= (2^n+1)^{2^{n-1}} \\ x_{n+1} &= (2^n+1)^{2^n} \end{aligned}$$

In particular, system (E_1)–(E_{n+10}) has at most one solution in non-negative integers.

Conversely, let $n \geq 2$ and let $x_1, \ldots, x_{n+11}$ be defined by (2), (3), (4), (15), (16), (17), (18), (19), (20), (21), (22), (23). In particular, since $n \geq 2$, Lemma 1 applied to $m = n$ guarantees that x_8 is an integer. The assumption $n \geq 2$ also implies that x_{n+5} is non-negative. It is straightforward to verify that these values of $x_1, \ldots, x_{n+11}$ satisfy (E_1)–(E_{n+10}).

Thus, the system (E_1)–(E_{n+10}) has $n+11$ variables and has a unique solution in non-negative integers. On the other hand, the largest value is

$$x_{n+1} = (2^n + 1)^{2^n} > 2^{n2^n} = 2^{n2^{-9} \cdot 2^{n+11-2}}.$$

In particular, if $n \geq 2^9$ then $x_{n+1} > 2^{2^{n+11-2}}$, which disproves the conjecture of the author.