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On the core of characteristic function games associated with exchange networks☆

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Abstract

An *exchange network* is a social system in which the actors gain valued resources from bilateral transactions, but an opportunity to negotiate a deal is given only to those pairs of actors whose positions are tied with each other in a fixed communication network. A transaction consists in a mutually agreed-on division of a resource pool assigned to a network line. An additional constraint imposed on such a network restricts the range of transaction sets which may happen in a single negotiation round to those consistent with a given "exchange regime." Under the *one-exchange regime* every actor is permitted to make no more than one deal per round. Bienenstock and Bonacich [Bienenstock, E.J., Bonacich, P., 1992. The core as a solution to exclusionary networks. Social Networks 14, 231–243] proposed to represent a one-exchange network with an *n-person game in characteristic function form*. The aim of this paper is to develop a *mathematical* theory of games associated with homogenous one-exchange networks (network homogeneity means that all lines are assigned resource pools of the same size). The focus is on the *core*, the type of solution considered most important in game theory. In particular, all earlier results obtained by Bonacich are re-examined and there is given a new *graph-theoretic* necessary and sufficient condition for the existence of nonempty core for the game representing a homogenous one-exchange network.

Keywords: Exchange network; Game theory; Game in characteristic function form; Assignment game; Core; Graph theory; Matching

☆ Two key theorems, 6.2 and 9.5, proven in this paper, were for the first time presented at the International Conference on Game Theory and Mathematical Economics, Warsaw, Stefan Banach International Mathematical Center, September 6–10, 2004.

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1. An exchange network as a mathematical object

The concept of an exchange network has been introduced by sociologists interested in studying social systems in which rational actors unequally benefit from bilateral transactions due to power inequality generated by the network structure. Following the publication of the special issue of *Social Networks* (1992) exchange networks have also become the object of purely mathematical investigations which are still going on (Willer, 1999; Bonacich, 1999; Sozański, 2004) along with experimental research.

We begin from defining an abstract *exchange network* as a mathematical object made up of three components.

(1) A *transaction opportunity graph* $G = (N, L)$

The *point set N* of *G* represents the positions occupied by *n actors* in the system. The actors can negotiate and conclude bilateral transactions with the restriction that a deal between the occupants of positions *P* and *Q* is permitted only if *PQ* is in the *line set L* of *G*. A *line* joining *P* and *Q*, written as *PQ* or *QP*, is a couple $\{P, Q\}$ of two distinct elements of *N*. We assume that *G* is *connected*, that is, any two distinct points are joined by a chain made up of lines in *L*. A collection of lines *M* is called a *chain* if it can be written as a sequence $u_1, \ldots, u_k$ such that u_i and u_{i+1} $(i = 1, \ldots, k - 1)$ have a common point. *M* is said to *join P with Q* if $P \in u_1$ and $Q \in u_k$.

(2) A *profit pool network C* over *G*

C is or a mapping which assigns to any line *PQ* in *L* a number $C_{PQ} > 0$ interpreted as the size of a *profit pool* to be divided between the occupants of positions *P* and *Q*. The term *transaction in a network line PQ* is referred to any pool split $x_{PQ} + x_{QP} = C_{PQ}$ agreed-on by *P* and *Q*.

(3) An *exchange regime* defined as a family $\mathcal{T}$ of subsets of *L*

Elements of $\mathcal{T}$ are called *transaction sets*. It is assumed that $\emptyset \in \mathcal{T}$ and for any line $PQ \in L$ there exists a transaction set $T \in \mathcal{T}$ such that $PQ \in T$. That is, the absence of a transaction is a "legal" outcome of the networkwide negotiation process and every line has a chance to be the locus of a transaction.

Transaction sets represent all configurations of bilateral agreements which may happen in a single negotiation round in accordance with a given rule which imposes limitations on the arrangement and number of transactions across the network. The term "exchange regime" and the germ of the idea elaborated here come from Friedkin (1992).

A transaction set *T* is called *maximal* is there is no *U* in $\mathcal{T}$ such that $T \subset U$ and $T \neq U$. That is, once for every line *PQ* in *T* the bargainers in positions *P* and *Q* have come to terms, no further transactions can be concluded, the negotiation round is over, and *P* and *Q* get their negotiated shares x_{PQ} and x_{QP} of C_{PQ}.

A *cumulative* exchange regime is defined by the following condition: for any $T \in \mathcal{T}$, if $U \subset T$, then $U \in \mathcal{T}$. Under a cumulative exchange regime, any one-line set $\{PQ\}$ is in $\mathcal{T}$ so that any two actors who have settled on a pool split can safely wait for the end of a round because their payoffs do not depend on transactions subsequently concluded elsewhere in the network.

An exchange regime is called *additive* if $T \cup U \in \mathcal{T}$ for any two point-disjoint transaction sets $T, U \in \mathcal{T}$. *T* and *U* are *point-disjoint* if there is no point *P* such that $PQ \in T$ and $PQ' \in U$ for some points *Q* and *Q*′.

To give an example of a cumulative and additive exchange regime, consider the *k-exchange regime* $\mathcal{T}_k(G)$ generated by the *k-exchange rule* which permits to every actor to make at most k transactions per round. Clearly, $T \in \mathcal{T}_k(G)$ if and only if $T \subset L$ and $\deg_T(P) \leq k$ for any $P \in N$ where $\deg_T(P)$ stands for the *degree* of P in the *subgraph* (N, T) of $G = (N, L)$, or the number of Qs such that $PQ \in T$.

Experimental research and formal theorizing has so far focused almost exclusively on *one-exchange networks*, or those with *one-exchange regime* $\mathcal{T}_1(G)$. $T \in \mathcal{T}_1(G)$ if and only if no two lines in T have a common endpoint. In graph theory (see Harary, 1969: Chapter 10), any subset T of L with this property is called a *matching* or an *independent set of lines*.

The structural parameter of $G = (N, L)$ known as the *line independence number* is defined by the formula

$$\beta_1(G) = \mathrm{Max}\{|T| : T \in \mathcal{T}_1(G)\}$$

where $|T|$ stands for the cardinality of T.

A matching T is called *optimal* if $|T| = \beta_1(G)$. We use the term "optimal matching" to avoid confusion with "maximal matching." In graph theory, the terms "maximal matching" and "maximum matching" are often used interchangeably to denote a matching of maximum cardinality. We need the concept of a maximal matching (maximal = maximal with respect to inclusion) to define the relations of exclusion and elementary power in the set of positions of a one-exchange network.

P can exclude Q if there exists a maximal matching T which covers P and does not cover Q. A collection of lines *M covers a point P* if P is an endpoint of a line in M. P is said to have *elementary power* over Q if P can exclude Q and Q cannot exclude P.

In this paper, we consider also the dual covering relation obtained by reversing the roles of points and lines, namely, we say that a subset S of the node set *N covers a line PQ* if $P \in S$ or $Q \in S$. The structural parameter known as the *point covering number*, noted $\alpha_0(G)$, is defined (Harary, 1969: Chapter 10) as the smallest number of nodes which cover all lines of G. Formally:

$$\alpha_0(G) = \mathrm{Min}\{|S| : S \text{ is a point cover of G}\}$$

where the term *point cover* is referred to any $S \subset N$ covering *all* lines in L.

The meaning of $\alpha_0(G)$ can be explained by the following "sociological" example. Suppose you want the actors in a social network to provide information on each *symmetric* tie. If each person is asked to report solely on those relations in which he or she is directly involved, then $\alpha_0(G)$ equals the *minimum* number of informers one must interview in order to gather the data on *all* symmetric network ties. To give an interpretation of $\beta_1(G)$, consider a task group in which the presence of a tie between two actors means that they can cooperate, and assume that any group member can cooperate with only one other member at a time. Then $\beta_1(G)$ is the maximum number of pairs who can work simultaneously.

2. Multiperson games in characteristic function form: the core

An *n-person game* in characteristic function form is formally defined (see, e.g. Owen, 1995: Chapter X) as a couple (N, v) made up of an n-element set N, called the set of *players*, and a mapping v, which assigns a real number $v(S)$ to any subset S of N. Subsets of N are called *coalitions*, v being termed *characteristic function*. The number assigned by v to S, named the *value* (or *worth*) of S, can be interpreted as the total payoff the members of S can win together independently of whether other players coordinate their actions within one big coalition $N - S$, few smaller coalitions, or play each on one's own account. It is convenient to assume that $v(\emptyset) = 0$.

Since all games we consider later in this paper have nonnegative individual and collective payoffs, we add the assumption that $v(S) \geq 0$ for any S and adopt the interpretation of the value of S as the amount of a divisible good the coalition S can secure for itself through concerted action. This quantity can be seen as the size of a "cake" from which each coalition member receives a portion provided that all players in S agree to coalesce and are able to settle on how to divide the "cake" among themselves. As regards the actors' goal-orientation, it is assumed that each player wants to gain as much as possible of the good in question and considers every potential coalition solely as a means to this end.

A game (N, v) is called *superadditive* if $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$ for any two disjoint coalitions S_1 and S_2. Superadditivity is usually included in the definition of an n-person game in characteristic function form. A somewhat weaker property is *monotonicity* defined by the condition: if $S_1 \subset S_2$, then $v(S_1) \leq v(S_2)$. In monotonic games, $v(S) \leq v(N)$ for every coalition S. Thus, the *grand coalition* N is given the largest cake to divide among its members.

Let $N = \{P_1, \ldots, P_n\}$. For any superadditive game, we define a *payoff vector* as a sequence of real numbers $x = (x_1, \ldots, x_n)$ such that

$$x_i \geq 0, \quad \text{for}\, i = 1, \ldots, n \text{ and } x_1 + \cdots + x_n \leq v(N)$$

where x_i is the number of resource units earned by player P_i at the end of the game. By assumption, all payoffs are nonnegative and can be jointly paid from the pool allocated to the grand coalition.

We say that a payoff vector x is *feasible for a coalition* S if $\sum(S, x) \leq v(S)$, where $\sum(S, x)$ stands for the total payoff of the members of S. By definition, every payoff vector is feasible for N.

We also assume that every time the game is played there arises a *coalition structure* $\{S_1, \ldots, S_k\}$, which is a partition of N into nonempty, pairwise disjoint subsets whose union is N. The coalition structures $\{\{P_1\}, \ldots, \{P_n\}\}$ and $\{N\}$ represent two simplest contrasting ways of playing the game: the actors fail to create any alliance or all ally to form the grand coalition.

Feasibility of x for a coalition structure $\{S_1, \ldots, S_k\}$ is defined by the condition: $\sum(S_j, x) = v(S_j)$ for $j = 1, \ldots, k$. The pair made up of a payoff vector and a coalition structure for which the payoff vector is feasible is called *payoff configuration*.

The condition by which we defined the feasibility of x for a coalition structure is stronger than the requirement that x is feasible for every S_j. However, the payoff vectors such that $\sum(S_h, x) < v(S_h)$ for some S_h need not be counted as outcomes of a superadditive game, since every payoff vector x with this property can always be replaced by an x' such that $\sum(S_j, x') = v(S_j)$ for all j, and $x'_i \geq x_i$ for every player P_i. To obtain x', each undistributed surplus $v(S_h) - \sum(S_h, x)$ is divided evenly among the members of S_h. Then, without any loss to the members of other coalitions, all players in S_h are better off. The inequality $\sum(N, x') \leq v(N)$ which is required of any payoff vector is met by x' in virtue of superadditivity, namely, $v(N) \geq v(S_1) + \cdots + v(S_k)$.

A payoff vector x is *feasible* if there exists at least one coalition structure for which x is feasible. Feasible payoff vectors form the most comprehensive "space of solutions" of a game in characteristic function form.

Mathematical analysis of such games takes into account both coalition formation and decisions on resource distribution, but usually focuses on one of two interrelated processes observed when the game is played. Coalition structures and payoff vectors are usually not tied with each other by a one-to-one correspondence. A payoff vector is often feasible for many coalition structures and a coalition structure usually admits of many payoff vectors feasible for it. For example, for every coalition structure $\{S_1, \ldots, S_k\}$ one can always define a feasible payoff vector by evenly splitting $v(S_j)$ among the members of S_j. Such a division of the resource pool could be considered plausible if the theory's objective would be to point out the "solutions" of the game that are in a sense "fair," or satisfy some formally stated postulates of distributive justice.

An alternative approach to "solving" n-person games assumes that the theoretically predicted payoffs should mirror unequal "strength" of particular actors. Although strategic skills are important when we study how the game is played by real or simulated players, a system-oriented game-theoretic analysis of power should focus on the differences entirely determined by the structure of a given game. The "power approach" to game solutions is not necessarily in conflict with "equity approach" if a "fair solution" means that a player's earning should reflect the "importance" of his cooperation for producing the collective good rather than his "right" to, say, a minimum reward.

An additional restriction which is usually imposed on the outcomes of a superadditive game is the condition of *individual rationality*:

$$x_i \geq v(\{P_i\}), \quad \text{for all } i.$$

This natural postulate means that no player will accept a lower payoff than the payoff he can safely gain by acting as a single actor coalition.

The *group* (*collective*) *rationality* condition is given by the equation

$$\sum (N, x) = v(N)$$

which means that x is feasible for the coalition structure $\{N\}$ made up of the grand coalition.

For any superadditive game (N, v) we have $v(N) \geq v(\{P_1\}) + \cdots + v(\{P_n\})$. Superadditive games for which $v(N) = v(\{P_1\}) + \cdots + v(\{P_n\})$ are called *inessential*. Then, for any S, $v(S)$ equals the sum of $v(\{Pi\})$ over all $P_i \in S$, which implies in turn that x such that $x_i = v(\{P_i\})$, $i = 1, \ldots, n$, is the only feasible, individually rational payoff vector. Since in these circumstances the players gain nothing by forming larger coalitions, the theory focuses on *essential* games, or those satisfying the inequality $v(N) > v(\{P_1\}) + \cdots + v(\{P_n\})$.

For essential games, the set of *imputations* (payoff vectors which meet the conditions of individual and group rationality) is usually too large so that further restrictions need to be imposed on theoretically predicted outcomes of the game.

The condition of *coalition rationality* has the form

$$\sum (S, x) \geq v(S), \quad \text{for all } S \subset N.$$

An intuitive meaning of this condition can be grasped by considering its negation, or the existence of a coalition S such that $\sum (S, x) < v(S)$. Clearly, the members of S will not be happy with such an outcome as they all can improve on their payoffs by forming the coalition and modifying x in such a way that each member of S will benefit. However, this may appear impossible without cutting the payoffs of the players from outside S. Therefore, it should not be a surprise that in many games there is no payoff vector satisfying the condition of coalition rationality. This condition is, in fact, very strong, as it asserts that no coalition can be prevented from using up its resource pool.

The set $\mathrm{Cr}(v)$ of coalitionally rational payoff vectors is called the *core* of the game (N, v). Coalition rationality implies that $x_i \geq v(\{P_i\})$, for $i = 1, \ldots, n$. Thus, all payoff vectors in the core satisfy the condition of individual rationality. Group rationality condition $\sum (N, x) = v(N)$ is also met because the inequality $\sum (N, x) \leq v(N)$ is assumed for every payoff vector and the inequality $\sum(N, x) \geq v(N)$ results from coalition rationality. Thus, the *core* can be equivalently defined as the *set of coalitionally rational imputations*.

Group rationality condition implies that every payoff vector in the core is feasible for the coalition structure $\{N\}$. Below we prove a theorem which specifies the range of coalition structures for which every payoff vector in $\mathrm{Cr}(v)$ is feasible.

We say that a *coalition structure* $\mathcal{S} = \{S_1, \ldots, S_k\}$ is *optimal* if it satisfies the condition

$$v(S_1) + \cdots + v(S_k) = v(N).$$

Theorem 2.1. *For any payoff vector x from the core of the game* (N, v), *x is feasible for a coalition structure* $\mathcal{S}$ *if and only if* $\mathcal{S}$ *is optimal.*

Proof. Assume that $x \in \mathrm{Cr}(v)$ and consider an optimal coalition structure $\mathcal{S} = \{S_1, \ldots, S_k\}$. We show that the optimality condition implies that x is feasible for $\mathcal{S}$. By applying the coalition rationality condition to every S_j and adding up the right and left hand sides of the respective inequalities, we arrive at the inequality $\sum_j \sum(S_j, x) \geq \sum_j v(S_j)$, or $\sum(N, x) \geq v(N)$. Since $\sum(N, x) = v(N)$, none of the inequalities $\sum(S_j, x) \geq v(S_j)$ can be sharp, which means that x is feasible for $\mathcal{S}$. The proof of the necessity of the condition $\sum_j v(S_j) = v(N)$ for the feasibility of x for the coalition structure $\{S_1, \ldots, S_k\}$ is even simpler: $v(N) = \sum(N, x) = \sum(S_1, x) + \cdots + \sum(S_k, x) = v(S_1) + \cdots + v(S_k)$ by group rationality and feasibility of x. □

Theorem 2.1 will help us to derive many results for the class of games we define in the next section after Bienenstock and Bonacich (1992) who were first to apply game theory to exchange networks.

3. The *n*-person game associated with an exchange network

The *n-person game associated with an exchange network* over $G = (N, L)$ with an exchange regime $\mathcal{T}$ is defined as $(N, v_{C,\mathcal{T}})$ where N – the set of network nodes – is taken as the set of players, and the value of the characteristic function $v_{C,\mathcal{T}}$ for any coalition $S \subset N$ is given by the formula

$$v_{C,\mathcal{T}}(S) = \mathrm{Max}\left\{\sum(T, C) : T \in \mathcal{T}; \text{if } PQ \in T, \text{then } P \in S \text{ and } Q \in S\right\}$$

where $\sum(T, C)$ stands for the sum of C_{PQ} over all PQ in T (we put $\sum(\emptyset, C) = 0$). The maximum of $\sum(T, C)$ is taken over all transaction sets T such that both endpoints of any line in T lie in S. Note that under any exchange regime

$$v_{C,\mathcal{T}}(\{P\}) = 0.$$

Assume that $\mathcal{T}$ is cumulative. Then $\{PQ\} \in \mathcal{T}$. Since $\{PQ\}$ is the only transaction set meeting the condition given in the definition of $v_{C,\mathcal{T}}(S)$ for $S = \{P, Q\}$, we have

$$v_{C,\mathcal{T}}(\{P, Q\}) = C_{PQ} \text{ for any } PQ \in L; \qquad v_{C,\mathcal{T}}(\{P, Q\}) = 0 \text{ for any } PQ \notin L.$$

The values of $v_{C,\mathcal{T}}$ for triplets and larger coalitions depend on a concrete exchange regime. For now, more specific results are available solely for one-exchange networks, Thus, we must conclude this section with only one general theorem.

Theorem 3.1. *If the exchange regime in an exchange network is cumulative and additive, then the associated game is superadditive and essential.*

The proof of Theorem 3.1 is an immediate consequence of the definitions, so it can be omitted.

By associating the characteristic function game $(N, v_{C,\mathcal{T}})$ with an exchange network $(N, L, C, \mathcal{T})$ we obtain a functor from the category of exchange networks to the category of superadditive and essential games.

Roughly speaking (to learn more, see Fararo, 1973: Chapter 19), a *concrete category* is a macro-object made up of: (1) a class of *mathematical objects*, each of them is a set endowed with a structure of a given type; (2) a class of *morphisms*, each morphism is a "structure-respecting" mapping of an object into another object.

The structuralist methodology of mathematics necessitates defining for every category at least the class of isomorphisms. In general, an *isomorphism* of two mathematical objects with *base sets* N_1 and N_2 and *structures* S_1 and S_2 of the same type is a 1–1 mapping π of N_1 onto N_2 which generates a 1–1 correspondence between S_1 and S_2. In particular, π is an isomorphism of two exchange networks $(N_1, L_1, C_1, \mathcal{T}_1)$ and $(N_2, L_2, C_2, \mathcal{T}_2)$ if: (1) π is an isomorphism of graphs (N_1, L_1) and (N_2, L_2), that is, for any $P, Q \in N$ such that $P \neq Q$, $PQ \in L_1$ if and only if $\pi(P)\pi(Q) \in L_2$; (2) there exists a number $s > 0$ such that for any line PQ in L_1 we have $C_2(\pi(P)\pi(Q)) = sC_1(PQ)$; (3) for any subset T of L_1, $T \in \mathcal{T}_1$ if and only if $\pi(T) \in \mathcal{T}_2$ where $\pi(T) = \{\pi(P)\pi(Q) : PQ \in T\}$.

Two characteristic function games (N_1, v_1) and (N_2, v_2) are isomorphic under π if there exists a scaling factor $s > 0$ such that $v_2(\pi(S)) = sv_1(S)$ for any $S \subset N$. Clearly, every isomorphism of two exchange networks is also an isomorphism of the games associated with them.

A *functor* is a macro-mapping from one category to another category which assigns objects to objects and morphisms to morphisms, and preserves the category's macro-structure: the operation of *morphism composition*. The functor from the category of exchange networks to the category of characteristic function games is defined for objects as the assignment $(N, L, C, \mathcal{T}) \to (N, v_{C,\mathcal{T}})$.

Let (N, v) be a superadditive, essential game with characteristic function v (recall that we assume that $v(S) \geq 0$ for any $S \subset N$). Does there exist an exchange network $(N, L, C, \mathcal{T})$ with cumulative and additive exchange regime $\mathcal{T}$ such that $v = v_{C,\mathcal{T}}$? For now we can only state an obvious necessary condition: $v(\{P\}) = 0$ for any $P \in N$; every S such that $v(S) > 0$ contains some points P and Q such that $v(\{P, Q\}) > 0$.

4. Generalized assignment games

Let $v_C = v_{C,\mathcal{T}_1(G)}$ denote the characteristic function of the n-person game associated with a one-exchange network over $G = (N, L)$. To compute $v_C(S)$ for any $S \subset N$, one needs to determine the maximum of $\sum(T, C)$ across all matchings which consist of lines having both endpoints in S. The subset of $\mathcal{T}_1(G)$ made up of these matchings coincides with $\mathcal{T}_1(G_S)$ where $G_S = (S, L_S)$ is the *subgraph* of $G = (N, L)$ *generated by S* (L_S stands for the subset of L made up of all lines with both endpoints in S).

A special case of the game (N, v_C) known as a *two-sided assignment game* is obtained by assuming that the transaction opportunity graph is *bipartite*, that is, N is the union of two nonempty, disjoint sets N_1 and N_2 such that every line in L has one point in N_1 and the other point in N_2.

Under a common economic interpretation, the members of N_1 are potential sellers and the members of N_2 are potential buyers of, say, used cars. Every seller (he) and every buyer (she) aims at maximizing his/her profit from a bilateral transaction with a member of the opposite class. A seller's profit is the difference between the negotiated price and the minimum price acceptable to him; a buyer's profit is the difference between maximum price she would pay for the car offered by a given seller and the negotiated price. If the minimum price of a seller P is lower than the maximum price of a buyer Q, then the two parties can bargain over the division of the difference between the two prices. Assuming that every seller has and every buyer needs only one car, one can model this simple market as a one-exchange network, and use its game representation to predict the outcome of the bargaining process.

Shapley and Shubik (1972) who were first to study two-sided assignment games proved that every game of the kind has a nonempty core (see Owen, 1995: pp. 221–223; Shubik, 1984: Chapter 8). They showed also that the core determines an "assignment" of buyers to sellers (hence the name "assignment game").

Bienenstock and Bonacich (1992) generalized the concept defined by Shapley and Shubik – by associating an n-person game with *any* exchange network. Their later papers (Bonacich and Bienenstock, 1995, 1997; Bonacich, 1998, 1999) provided further results on the existence and shape of the core for the games associated with one-exchange networks. I propose to call these games *generalized assignment games*. The term "the Bienenstock–Bonacich game" will be used interchangeably.

This paper offers a graph-theoretic characterization of the core for the games associated with *homogenous* one-exchange networks, or those in which every line is assigned a profit pool of the same size, that is, for any $PQ \in L$, $C_{PQ} = C_0$ for some $C_0 > 0$. Since the context makes confusion impossible, the letter C will be used to denote both the profit pool matrix and the constant pool size. The choice of $C > 0$ affects only the scale for measuring the value of each coalition.

It is not difficult to verify that the characteristic function of a homogenous generalized assignment game is given by the following formula

$$v_C(S) = C\beta_1(G_S)$$

which holds for any subset S of N. In particular, we have $v_C(N) = C\beta_1(G)$.

All terms introduced in Section 2 for any characteristic function game can be applied to generalized assignment games. However, if such a representation of a one-exchange network is regarded as a tool for analyzing the network itself, then one must study, first of all, those "solutions" of the network game which are compatible with the assumption that the networkwide payoff allocation should arise from *two-party* transactions.

Consider the homogenous TRIAD network with $L = \{A_1A_2, A_1A_3, A_2A_3\}$. The payoff vector x such that $x_i = \frac{1}{3}C$ for $i = 1, 2, 3$ is feasible only for the coalition structure $\{N\}$, but the payoff configuration $(x, \{N\})$ can be accepted as an outcome of an experimental game only if the players – having collected their "official" payoffs from bilateral transactions – are allowed to divide evenly among themselves all the money they won together. However, a "legal" outcome of the game associated with the TRIAD network must take the form $(0, a, C - a)$, $(a, 0, C - a)$, $(a, C - a, 0)$. The first payoff vector is feasible for the coalition structure $\{\{A_1\}, \{A_2, A_3\}\}$, the second is feasible for $\{\{A_2\}, \{A_1, A_3\}\}$, and the third for $\{\{A_3\}, \{A_1, A_2\}\}$.

The *network-oriented* approach to games associated with exchange networks prompts the following definition. A payoff vector x is said to be *feasible for a matching* T if x is feasible for the coalition structure determined by T, that is, the coalition structure made up of two-player coalitions coinciding with lines in T and of single-player coalitions corresponding to points not covered by T. Similarly, we call x a *network-feasible* payoff vector if x is feasible for a matching T.

A consequence of network feasibility is that P_i's payoff x_i in a game associated with a one-exchange network can be identified with the number of profit points P_i gets in a transaction with one of its neighbors in G, formally, $x_i \leq C_{P_iP_j}$ for some node P_j such that $P_iP_j \in L$.

Note that a network-feasible payoff vector can be feasible for more than one matching. For example, in the homogenous BOX network with four nodes A_1, A_2, A_3, A_4 and four lines which form a cycle, any payoff vector x such that $x_1 = a$, $x_2 = C - a$, $x_3 = a$, $x_4 = C - a$ is feasible for two matchings: $\{A_1A_2, A_3A_4\}$ and $\{A_2A_3, A_1A_4\}$.

To coordinate and simplify the notation of network and game quantities, we can label the nodes of an exchange network and the players with the same integers $1, \ldots, n$, and represent the profit pool network C over $G = (N, L)$ as an $n \times n$ *matrix* $C = (C_{ij})$ where $C_{ij} = C_{P_iP_j}$ for $P_iP_j \in L$ and $C_{ij} = 0$ for $P_iP_j \notin L$. The C matrix uniquely determines the *adjacency matrix* $G = (G_{ij})$ of the transaction opportunity graph: $G_{ij} = 1$ if $C_{ij} > 0$, $G_{ij} = 0$ if $C_{ij} = 0$.

A matching T is called *network-optimal* if $\sum(T, C) = v_C(N)$. In a homogenous one-exchange network, $v_C(N) = C\beta_1(G)$ and $\sum(T, C) = C|T|$ for every matching T, so that T is network-optimal if and only if $|T| = \beta_1(G)$, that is, network-optimality reduces to optimality defined in Section 1.

The theorem which follows shows how network-optimal matchings are related to the core of a generalized assignment game.

Theorem 4.1. *If a payoff vector x is in the core of a generalized assignment game, then x is feasible for a matching T if and only if T is network-optimal.*

Proof. Let $\{S_1, \ldots, S_k\}$ be the coalition structure generated by a matching T, that is, $S_j = \{P, Q\}$ for some PQ in T or $S_j = \{P\}$ for some P. Since $v_C(\{P, Q\}) = C_{PQ}$ for any PQ in T and $v_C(\{P\}) = 0$ for any P, we have $v_C(S_1) + \cdots + v_C(S_k) = \sum(T, C)$. As a consequence, the conditions $\sum(T, C) = v_C(N)$ and $v_C(S_1) + \cdots + v_C(S_k) = v_C(N)$ are equivalent, but the latter condition was given in Theorem 2.1 as necessary and sufficient for the feasibility of $x \in \mathrm{Cr}(v_C)$ for $\{S_1, \ldots, S_k\}$. □

Theorem 4.1 specifies for the game (N, v_C) a class of payoff configurations which can be considered as admissible outcomes of the groupwide negotiation process in the underlying one-exchange network. They have the form (x, T) where $x \in \mathrm{Cr}(v_C)$ and T is a network-optimal matching. The feasibility of x for T means that $x_i + x_j = C_{ij}$ for any line P_iP_j in T, and $x_k = 0$ for any point P_k not covered by T. That is, P_i and P_j split the pool assigned to P_iP_j between themselves, while P_k gains nothing for failure to find a partner for a deal.

5. Properties of the core in a generalized assignment game

"The core is, perhaps, the most intuitive solution concept in cooperative game theory. Nevertheless, quite frequently it is pointed out that it has several shortcomings, some of which are given below: (1) The core of many games is empty [. . .]. (2) In many cases the core is too big [. . .]. (3) In some examples the core is small but yields counter-intuitive results." (Peleg, 1992: p. 398). As we shall see, all three shortcomings of the core mentioned by Peleg occur in generalized assignment games. "Corelessness" is the most serious trouble insofar as one expects from the theory to predict the outcomes of exchange process.

TRIAD is the smallest one-exchange network for which the existence of the core for the associated game depends on profit pool sizes assigned to lines A_1A_2, A_1A_3 and A_2A_3.

Assume that $C_{12} = \mathrm{Max}\,\{C_{12}, C_{13}, C_{23}\}$ so that $v_C(N) = C_{12}$. Let $x = (x_1, x_2, x_3)$ be a payoff vector. If $x \in \mathrm{Cr}(v_C)$, then x is feasible for the network-optimal matching $\{A_1A_2\}$, which means that $x_1 + x_2 = C_{12}$ and $x_3 = 0$. By applying the coalition rationality condition to $\{A_1, A_3\}$ and $\{A_2, A_3\}$, we arrive at the inequalities $x_1 \geq C_{13}$ and $x_2 \geq C_{23}$ which imply in turn that $C_{12} \geq$

$C_{13} + C_{23}$. Therefore, the latter inequality is the necessary condition of the existence of nonempty core for v_C. Clearly, it is a sufficient condition as well. Since it is not met if $C_{12} = C_{13} = C_{23}$, the homogenous TRIAD network is the simplest Bienenstock–Bonacich game with empty core.

The game associated with a one-exchange network over the 3-CHAIN or 2-STAR graph of the form B_1–A–B_2 has a nonempty core under any weights assigned to lines AB_1 and AB_2. It is not difficult to show that the core reduces to the payoff vector such that $x_{B_1} = x_{B_2} = 0$ and $x_A = \text{Max}\,\{C_{AB_1}, C_{AB_2}\}$. Such an extreme imbalance of benefits is unlikely to occur in the games played by experimental subjects.

Although the core may be in many cases inadequate for making *quantitative* payoff predictions, we show later in this paper that it can be used to draw a *qualitative* distinction between "strong" and "weak" variety of *game* power, analogous to the distinction made within the theory of *exclusionary* power.

Let P be a node of a one-exchange network over $G = (N, L)$. We say that P is *excludable* if there exists a maximal matching T which does not cover P. P is called *nonexcludable* if it is covered by *all* maximal matchings. A nonexcludable P will always find a partner provided that a negotiation round continues until all transaction opportunities permitted by the one-exchange regime are used up.

The game-theoretic counterpart of excludability is introduced by the following definition: P is *game-excludable* if there exists a network-optimal matching T which does not cover P. Let us recall in this connection the parameter defined by Cook et al. (1983), called by them "Reduction in Maximum Flow" (RMF). RMF can be redefined in game-theoretic language by the formula: $\text{RMF}(P) = v_C(N) - v_C(N - \{P\})$. P is game-excludable if and only if $\text{RMF}(P) = 0$.

Since every optimal matching is maximal, every game-excludable point is excludable. The converse is not true, as there exist excludable points that are not game-excludable. To give an example, consider the homogenous one-exchange network whose transaction opportunity graph has the form B_1–A_1–A_2–B_2 justifying the nickname 4-CHAIN. Points B_1 and B_2 are excludable because they are not covered by the maximal matching $\{A_1A_2\}$, yet they are game-nonexcludable because they are covered by the matching $\{A_1B_1, A_2B_2\}$ which is the only optimal matching in this network.

Theorem 5.1 implies that game-excludable positions gain nothing if the network game ends up with an outcome in the core.

Theorem 5.1. *If a payoff vector x is feasible for all network-optimal matchings*, then $x_i = 0$ *for every game-excludable P_i.*

The proof of Theorem 5.1 is straightforward. Let T be a network-optimal matching which does not cover P_i. The feasibility of x for T implies that $x_i = v_C(\{P_i\}) = 0$. In particular, if $x \in \text{Cr}(v_C)$, then $x_i = 0$ for any game-excludable P_i.

The property of game-excludability can be defined for any superadditive game (N, v) by the following statement: a *coalition $S \subset N$* is *game-excludable* if $v(N) = v(N - S)$, in other words, the group does not need the participation of the members of S to achieve the maximum possible collective profit. A *player $P \in N$* is said to be *game-excludable* if coalition $\{P\}$ is game-excludable. The generalization of Theorem 5.1 takes the following form: *If a payoff vector x is feasible for all optimal coalition structures in a superadditive game* (N, v) *and coalition S is game-excludable*, *then* $x_i = 0$ *for all* P_i in S. The prove this fact, notice that $v(N) \geq v(N - S) + v(S)$, $v(N) \geq v(N) + v(S)$, $0 \geq v(S)$, and $v(S) = 0$, in virtue of superadditivity, game-excludability of S, and the assumption, made in Section 2, that a characteristic function takes nonnegative values. Since $v(S) + v(N - S) = v(N)$, the coalition structure $\{S, N - S\}$ is optimal. The feasibility of x for this coalition structure means that $\sum(S, x) = v(S) = 0$. Since $x_i \geq 0$, by assumption, there must be $x_i = 0$ for any $P_i \in S$.

There are many homogenous one-exchange networks in which all positions are game-excludable, the homogenous TRIAD network being the smallest example. Theorem 5.1 implies that the games associated with such networks are coreless. Indeed, if every position P_i is game-excludable and $x \in \mathrm{Cr}(v_C)$, then $x_i = 0$ for all i so that $\sum(N, x) = 0$, which contradicts the group rationality condition $\sum(N, x) = v_C(N) > 0$.

There exist coreless homogenous one-exchange networks containing game-nonexcludable nodes. The smallest example can be drawn as a "triangle with a two-line tail" (the chain *A*–*C*–*D* is added to TRIAD with nodes labeled A, B_1, B_2). In this network, node *C* is nonexcludable, so it is also game-nonexcludable.

In coreless generalized assignment games, game-excludable players are not always doomed to being "exploited" by their game-nonexcludable neighbors. Two game-excludable players can be connected with each other, which gives them the opportunity to evenly split the pool between themselves. However, if a Bienenstock–Bonacich game has a nonempty core, the situation of game-excludable players is much worse as their *only* potential transaction partners are game-nonexcludble. This results from the following theorem.

Theorem 5.2. *If a generalized assignment game has a nonempty core, then no two game-excludable nodes are tied with each other in the network.*

Proof. Assume that there exists a payoff vector x in $\mathrm{Cr}(v_C)$. Theorem 5.1 implies that $x_i = 0$ and $x_j = 0$ for any two game-excludable nodes P_i and P_j. If line P_iP_j were in G, then $0 = x_i + x_j \geq v_C(\{P_i, P_j\}) = C_{ij} > 0$, which is a contradiction. □

The two subsequent theorems were first demonstrated by Bonacich and Bienenstock (1995, 1997). Theorem 5.3 simplifies the verification of the coalition rationality condition, namely, it suffices to consider only the dyadic coalitions in which utility transfer is possible. If all connected dyads are coalitionally rational, then so is every larger coalition.

Theorem 5.3. *An n-dimensional vector x is in the core of the game (N, v_C) associated with the one-exchange network C over $G = (N, L)$ if and only if x satisfies the following three conditions*:

(1) $x_i \geq 0$, *for all* $P_i \in N$;
(2) $x_i + x_j \geq C_{ij}$, *for all* $P_iP_j \in L$;
(3) $x_1 + \cdots + x_n = v_C(N)$.

Proof. The only thing we have to demonstrate is that condition (2) implies the rationality of any $S \subset N$ such that $|S| > 2$. Let T be a matching in G_S such that $\sum(T, C) = v_C(S)$ and S' be a subset of S made up of points covered by T. Then $\sum(S, x) = \sum(S', x) + \sum(S - S', x) \geq \sum(S', x)$ because $\sum(S - S', x) \geq 0$ in virtue of condition (1). Clearly, $\sum(S', x)$ is the sum of $x_i + x_j$ over all $P_iP_j \in T$. Since $x_i + x_j \geq C_{ij}$ by condition (2), we arrive at the inequality $\sum(S', x) \geq \sum(T, C)$, which implies that $\sum(S, x) \geq v_C(S)$. □

Condition (3) in Theorem 5.3 can be replaced by a weaker condition

$$(3')\, x_1 + \cdots + x_n \leq v_C(N)$$

because conditions (1) and (2) imply that $x_1 + \cdots + x_n \geq v_C(N)$. Conditions (1) and (3′) amount to the assumption that x is a payoff vector. Thus, Theorem 5.3 can also be stated as the following equivalence: *a payoff vector x is in the core if and only if $x_i + x_j \geq C_{ij}$ for any network line P_iP_j.*

To solve a system of inequalities is a more difficult task than to solve a system of equations. Therefore, the problem of whether the network game has a nonempty core and what payoff vector are in there would be easier to cope with if one could replace with equations at least some of the inequalities of the form $x_i + x_j \geq C_{ij}$. It turns out that equations $x_i + x_j = C_{ij}$ and sharp inequalities $x_i + x_j > C_{ij}$ occur in two distinct types of lines in L, network-optimal and network-suboptimal.

A line PQ in L is called *network-optimal* if $PQ \in T$ for some network-optimal matching T. Let L^0 denote the subset of L which consists of network-optimal lines. Network-suboptimal lines are the elements of $L - L^0$. Since for homogenous networks the terms "network-optimal matching" and "optimal matching" mean the same, the term "optimal line" will be used instead of "network-optimal line" in this case.

Theorem 5.4. *If a payoff vector x is in the core of the game* (N, v_C), *then* $x_i + x_j = C_{ij}$ *for every network-optimal line* $P_i P_j$.

Theorem 5.4 which is a trivial consequence of Theorem 4.1 shows which lines are most likely to be used for transactions if the game associated with a one-exchange network C has a nonempty core.

6. A graph-theoretic criterion for the existence of nonempty core in a homogenous Bienenstock–Bonacich game with one game-component

My mathematical research on the Bienenstock–Bonacich games began in 1996 when I found the term "line-core" in Chapter 10 of Harary's *Graph Theory* (1969). Although at first sight the game-theoretic core and graph-theoretic line-core seemed to have nothing to do with each other, a closer look at the two concepts led me to the discovery of Theorem 6.1. Meanwhile, Bonacich and Bienenstock (1997) arrived at the idea of a decomposition of a one-exchange network into parts that are called "game-components" later in my paper. Five years after the proof Theorem 6.1 I resumed work on Bienenstock–Bonacich games and proved Theorem 6.2 which offers a graph-theoretic criterion for the existence of nonempty core for all homogenous one-exchange networks having exactly one "game-component."

Henceforth, we consider solely *homogenous* one-exchange networks. Since the choice of the constant C is inessential, we put $C = 1$, and write v_G instead of v_C to mark that the characteristic function is now determined uniquely by $G = (N, L)$ according to the formula: $v_G(S) = \beta_1(G_S)$.

We show in this section that the existence of a nonempty core for the game (N, v_G) depends on two structural parameters of G, defined in Section 1, line independence number $\beta_1(G)$ and point covering number $\alpha_0(G)$.

Let T be an optimal matching. To cover all lines in L, one needs to pick at least one point from each line in T, and possibly add some points taken from the lines in $L - T$. Therefore, $\alpha_0(G) \geq \beta_1(G)$. If $\alpha_0(G) = \beta_1(G)$, G is said to have a line-core (the *line-core* itself is defined as the union of all matchings T such that $|T| = \alpha_0(G)$; see Harary, 1969: p. 98).

We are now in a position to prove the following *sufficient* condition for the existence of a nonempty core for the Bienenstock–Bonacich game.

Theorem 6.1. *If* $\beta_1(G) = \frac{1}{2}n$ *or* $\beta_1(G) = \alpha_0(G)$, *then the game associated with a homogenous one-exchange network over G has a nonempty core.*

Proof. If $v_G(N) = \beta_1(G) = \frac{1}{2}n$, we put $x_i = \frac{1}{2}$ for all i. Since all three conditions in Theorem 5.3 are met, the payoff vector so defined is in the core. Assume now that $\beta_1(G) = \alpha_0(G)$. Let S be a minimum point cover for G, that is, $|S| = \alpha_0(G)$. Put $x_i = 1$ for all $P_i \in S$ and $x_i = 0$ for all $P_i \in N - S$. Let us show that $x \in \mathrm{Cr}(v_G)$. Clearly, $x_i \geq 0$ for all $P_i \in N$ so that condition (1) in Theorem 5.3 is satisfied. Since $\sum(N, x) = |S| = \alpha_0(G) = \beta_1(G) = v_G(N)$, condition (3) holds as well. To complete the proof, one needs to show that $x_i + x_j \geq 1$ for every line $P_iP_j \in L$. Since S is a point cover for G, for every line $P_iP_j \in L$ we have $P_i \in S$ or $P_j \in S$. Therefore, $x_i = 1$ or $x_j = 1$, and $x_i + x_j \geq 1$. □

Theorem 6.1 gives two different sufficient conditions for the existence of the core: $\beta_1(G) = \frac{1}{2}n$ and $\beta_1(G) = \alpha_0(G)$. The complete 4-node graph K_4 satisfies the first condition but not the second ($\beta_1(K_4) = 2$ and $\alpha_0(K_4) = 3$). The 3-CHAIN (2-STAR) graph B_1–A–B_2 satisfies the second condition but not the first. In many graphs with even number of nodes both hold true, the DYAD A_1–A_2 being the simplest example.

The union $\beta_1(G) = \frac{1}{2}n$ *or* $\beta_1(G) = \alpha_0(G)$ of two conditions is not necessary for the existence of the core. However, there are only eight one-exchange networks with at most eight nodes for which both conditions are not met and the associated game has a nonempty core. These networks are obtained by connecting the central node in a 2-STAR or 3-STAR with one or more nodes in K_4 (a k-STAR is a graph whose line set has the form $\{AB_i : i = 1, \ldots, k\}$).

Theorem 6.1 implies that every *homogenous* two-sided assignment game has a nonempty core. To demonstrate this, one needs König's theorem (see Harary, 1969: Theorem 10.2) which states that $\beta_1(G) = \alpha_0(G)$ for any bipartite graph G. However, Shapley and Shubik (1972) proved a stronger existence theorem than our corollary. Their theorem applies to *all* two-sided assignment games, not only homogenous.

Following (Bonacich, 1998) we consider for every graph $G = (N, L)$ its subgraph $G^0 = (N, L^0)$ whose line set L^0 consists of all optimal lines in G. If G^0 is *connected*, that is, any two distinct points are joined by a chain made up of lines in L^0, we say that the one-exchange network over G is *game-indecomposable*.

Theorem 6.2. *If the one-exchange network over G is game-indecomposable, then the core of the associated game* (N, v_G) *is not empty if and only if* $\beta_1(G) = \frac{1}{2}n$ *or* $\beta_1(G) = \alpha_0(G)$.

Proof. We have already proven (Theorem 6.1) that condition $\beta_1(G) = \frac{1}{2}n$ *or* $\beta_1(G) = \alpha_0(G)$ suffices for the existence of a nonempty core for any G. Assume now that G is game-indecomposable. To prove necessity, assume that $x \in \mathrm{Cr}(v_G)$ for some x. Consider an optimal matching T in G. Suppose that $\beta_1(G) < \frac{1}{2}n$. The proof of Theorem 6.2 will be completed if we show that $\alpha_0(G) = \beta_1(G)$. Since $|T| < \frac{1}{2}n$, there exists a point P_i not covered by T. Theorem 5.1 implies that $x_i = 0$. The connectedness of $G^0 = (N, L^0)$ means that for every $P_j \neq P_i$ there exists a chain from P_i to P_j made up of optimal lines. It follows from Theorem 5.2 that the x values assigned to the endpoints of any line in this chain sum up to 1. As a consequence, $x_h = 0$ or $x_h = 1$ for all $P_h \in N$.

We define the set of points S by means of the condition $P_j \in S$ if $x_j = 1$ and $P_jP_k \in T$ for some P_k. Let S' be the set of the remaining points which occur in lines in T. Since for every line $P_jP_k \in T$ we have $x_j + x_k = 1$, either P_j is in S and P_k is in S' or conversely. Therefore, $|S| = |T| = \beta_1(G)$. To derive the conclusion that $\alpha_0(G) = \beta_1(G)$, we must only show that S is a point cover for G.

Suppose for the indirect proof that $P_i \notin S$ and $P_j \notin S$ for some $P_iP_j \in L$. $P_i \notin S$ implies that $P_i \in S'$ or $P_i \in N - (S \cup S')$. If $P_i \in S'$, then $x_i = 0$. If $P_i \in N - (S \cup S')$, then $x_i = 0$ as well, which results from Theorem 5.1 and the fact that P_i is game-excludable as a point not covered by the optimal matching T. Similarly, $P_j \notin S$ implies that $x_j = 0$. The conclusion that $x_i + x_j = 0$

contradicts the inequality $x_i + x_j \geq 1$ which must hold for every $P_i P_j \in L$ according to condition (2) in Theorem 5.3. The proof of Theorem 6.2 is completed. □

7. The core as a solution of a system of linear equations

In this section, we analyze a linear-algebraic approach which was proposed by Bonacich (1998, 1999) to help verify if a game-indecomposable homogenous one-exchange network has a nonempty core. Consider the following system of linear equations:

$$x_i + x_j = 1, \quad \text{for all optimal lines } P_i P_j \text{in} G; \tag{1}$$

$$x_1 + \cdots + x_n = \beta_1(G). \tag{2}$$

Let m denote the number of optimal lines in G, that is, the number of lines in G^0. If the game (N, v_G) has a nonempty core, then the system of $m + 1$ equations made up of m Eq. (1) and Eq. (2) has at least one solution. To apply a well known criterion for the solvability of a system of linear equations, Bonacich represents the subsystem 1 by the matrix equation

$$Bx = \mathbf{1} \tag{1'}$$

where **1** stands for the column vector made up of 1s, and B is the transpose of the incidence matrix of G^0. Recall that the *incidence matrix* (see Harary, 1969: Chapter 13) of an undirected graph is a 0–1 rectangular matrix whose rows and columns correspond to points and lines of the graph, and each ij entry equals 1 if and only if ith point is one of the endpoints of jth line.

Being the transpose of the incidence matrix of G^0, B has m rows and n columns. By multiplying B by an n-dimensional column vector x representing the payoffs of n players, one gets an m-dimensional column vector Bx. If x satisfies all m Eq. (1), then all coordinates of Bx are 1s.

To represent Eqs. (3) and (2) by one matrix equation, Bonacich introduces matrix A which is obtained from B by adjoining $(m + 1)$th row all made up of 1s. When this last row of A is multiplied by the column vector $x \in \mathrm{Cr}(v_G)$, the result is $\beta_1(G)$. Therefore, Eqs. (3) and (2) are together equivalent to

$$Ax = w \tag{3}$$

where w is the column vector whose first m coordinates are 1, and $(m + 1)$th coordinate is $\beta_1(G)$. It is clear that all payoff vectors in the core of (N, v_G) satisfy the matrix Eq. (3). However, not all solutions of this equation are in the core.

Consider the DBOX graph with four points A_1, A_2, B_1, B_2, and five lines of which four, A_1B_1, A_1B_2, A_2B_1, A_2B_2 can be drawn as a box, and the fifth one, A_1A_2, as a diagonal joining A_1 and A_2. For this graph, any solution of (3) has the form: $x_{A_1} = x_{A_2} = a$, and $x_{B_1} = x_{B_2} = 1 - a$, but only the payoff vectors with $a \geq \frac{1}{2}$ are in the core of the Bienenstock–Bonacich game.

Thus, the problem of how to dispose of inequalities in the examination of core nonemptiness, should be stated as follows. Assume that equation $Ax = w$ is solvable. Does the set of its solutions contain a nonnegative vector x which satisfies also the inequality $x_i + x_j \geq 1$ for all suboptimal lines? If the answer to this question is positive, then one can use the methods of linear algebra to verify if $\mathrm{Cr}(v_G) \neq \emptyset$.

Bonacich recommends to compute the rank of A and the rank of the matrix $A|w$ obtained from A by adding w as $(n + 1)$th column. Recall that the *rank* of a rectangular matrix is the maximum number of linearly independent columns, or equivalently, the maximum number of linearly independent rows. Eq. 3 has at least one solution if and only if the rank of $A|w$ equals the rank of A.

Theorem 7.1. *The game associated with a game-indecomposable homogenous one-exchange network has a nonempty core if and only if the matrix equation $Ax = w$ has at least one solution.*

An equivalent formulation of Theorem 7.1 is that the existence of a payoff vector feasible for all optimal matchings is a sufficient and necessary condition for a Bienenstock–Bonacich game to have a nonempty core. In order to check if the feasibility condition is met, no inequalities need to be solved whatsoever.

The proof of Theorem 7.1 we provide here patches the gaps in the analysis done by Bonacich (1998, 1999). It is longer and less trivial than other proofs given in this paper. In particular, we will employ the following theorem which is a powerful tool in the theory of matchings.

Any bipartite graph $G = (N, L)$ with $N = N_1 \cup N_2$ admits a matching which covers all points in N_1 if and only if $|\Gamma(S)| \geq |S|$ for any $S \subset N_1$.

$\Gamma(S)$ stands here for the set of points which are adjacent to points in S, formally, $\Gamma(S) = \{Q \in N_2 : PQ \in L \text{ for some } P \in S\}$. A simple proof of this theorem, usually referred to as the König–Hall theorem, can be found in Ore's book (1963).

Proof of Theorem 7.1. Assume that equation $Ax = w$ admits of at least one solution x. We distinguish two cases depending on whether G^0 is or is not bipartite. According to König's theorem (see Harary, 1969: Theorem 2.4) a graph is bipartite if and only if every cycle in it (a *cycle* is a chain which joins P with P for some P) consists of an even number of lines (acyclic graphs, or those containing only 0-cycles, also satisfy this condition).

If G^0 is not bipartite, then L^0 contains a cycle of an odd length, say, of length 3, P_{i_1}–P_{i_2}–P_{i_3}–P_{i_1}. The equations $x_{i_1} + x_{i_2} = 1$, $x_{i_2} + x_{i_3} = 1$, $x_{i_3} + x_{i_1} = 1$ imply that $x_{i_1} = \frac{1}{2}$, but then $x_j = \frac{1}{2}$ for every j due to connectedness of G^0. Thus, if G^0 is not bipartite, then the equation $Ax = w$ has the unique solution which is in the core.

If G^0 is bipartite, then N can be partitioned into two disjoint nonempty subsets N_1 and N_2 with n_1 and n_2 points ($n_1 + n_2 = n$). Any solution x of equation $Bx = \mathbf{1}$ is then determined by an a such that $x_i = a$ for all P_i in N_1 and $x_j = 1 - a$ for all P_j in N_2. An additional constraint on a is imposed by the group rationality condition $x_1 + \cdots + x_n = \beta_1(G)$ which takes the form $\beta_1(G) = n_1 a + n_2(1 - a) = n_2 + (n_1 - n_2)a$. We can label the two sets which form the partition of N in such a way that $a \geq 1 - a$, or $a \geq \frac{1}{2}$. If $n_1 > n_2$, then $\beta_1(G) > n_2 + \frac{1}{2}(n_1 - n_2) = \frac{1}{2}n$, which is impossible. Therefore, $n_1 \leq n_2$. If $n_1 = n_2$, then $\beta_1(G) = n_2 = \frac{1}{2}n$, and the group rationality condition holds for every a, so one can put $a = \frac{1}{2}$ to get a vector meeting all core conditions.

Assume in turn that $n_1 < n_2$, which implies that a is determined uniquely from the formula $a = (n_2 - \beta_1(G))/(n_2 - n_1)$. We have already mentioned that the collection of equations $x_i + x_j = 1$ for all $P_i P_j \in L^0$ *plus* the equation $x_1 + \cdots + x_n = \beta_1(G)$ is equivalent to the feasibility of x for all optimal matchings. This fact will allow us to derive the identity $\beta_1(G) = n_1$. Clearly, $\beta_1(G) = \beta_1(G^0) \leq n_1$. If $\beta_1(G) < n_1$, then there is an optimal matching T such that $|T| < n_1$, which implies in turn the existence of a game-excludable node P_i in N_1. According to Theorem 5.1, $x_i = 0$, but we already know that $x_i = a \geq \frac{1}{2}$. Thus, $\beta_1(G) = n_1$, which implies that $a = 1$.

To sum up, we have demonstrated that for bipartite G^0 the equation $Ax = w$ of which we assumed to have *a* solution admits *the* 0–1 solution. It is in the core provided that the inequality

$x_i + x_j \geq 1$ holds for every line $P_iP_j \in L - L^0$. Since this inequality holds true for any line with at least one point in N_1 (recall that $x_i = 1$ for any P_i in N_1), to complete the proof, we have to examine the case of a suboptimal line with both endpoints in N_2. We show that L may not contain such a line, which will be derived from two facts:

(1) if P_i and P_j are in N_2, and P_i is game-excludable, then $P_iP_j \notin L$;
(2) Every P_i in N_2 is game-excludable.

Proof of (1). Suppose that $P_iP_j \in L$. Since P_i is game-excludable, there exists an optimal matching T which does not cover P_i. If T does not cover P_j, then line P_iP_j could be added to T so that T would not be optimal. If T covers P_j, then the line which covers P_j could be replaced with P_iP_j to obtain an optimal matching T' containing P_iP_j. Therefore, $P_iP_j \in L^0$, $x_i + x_j = 1$, but $x_i = 0$ and $x_j = 0$, since both points are in N_2.

Before we prove (2), observe that every P_i in N_1 is game-nonexcludable.

Proof of (2). First, we show that every k-element subset S of N_1 has in total at least $k + 1$ neighbors in N_2, that is, $|\Gamma(S)| \geq k + 1$. Clearly, $|\Gamma(S)| \geq k$. Assume that $k = n_1$. Then we have, $S = N_1$, $\Gamma(S) = N_2$, and $n_2 > n_1$. Assume in turn that $k < n_1$. By connectedness of G^0, there exists a line P_iP_j in L^0 such that $P_i \in N_1 - S$ and $P_j \in \Gamma(S)$. Let T be an optimal matching in G containing P_iP_j. Suppose that $|\Gamma(S)| = |S|$. T cannot cover all points in S, since one point in $\Gamma(S)$ was matched with a point outside S. Therefore, S must contain a game-excludable node, which is a contradiction. The condition $|\Gamma(S)| > |S|$ for any $S \subset N_1$ implies that every set of k points in N_1 must have in total at least k neighbors in the graph $G' = G^0 - \{P_j\}$ obtained from G^0 by deleting a point P_j in N_2. The König–Hall theorem implies the existence of a matching T in G' covering all points in N_1. Since T is also an optimal matching in G and it does not cover P_j, P_j is game-excludable, which completes the proof of (2) and the proof of Theorem 7.1. □

Another characterization of indecomposable networks with nonempty core was also inspired by Bonacich's conjecture (personal communication, 1997).

Theorem 7.2. *If the subgraph G^o of G made up of optimal lines in G is connected, then the game associated with G has a nonempty core if and only if G^0 is bipartite or $\beta_1(G) = \frac{1}{2}n$.*

Proof. To prove necessity, assume that the game (N, v_G) has a nonempty core and G^0 is *not* bipartite. Then the unique payoff vector in $\mathrm{Cr}(v_G)$ has the form $x_i = \frac{1}{2}$, $i = 1, \ldots, n$ (see the proof of Theorem 7.1), which implies that $\beta_1(G) = \frac{1}{2}n$.

To prove sufficiency, assume that $\beta_1(G) < \frac{1}{2}n$ and G^0 is bipartite. Then $\beta_1(G^0) = \alpha_0(G^0)$ in virtue of König's theorem. On applying Theorem 6.2, we conclude that the game associated with G^0 has a nonempty core. Let $x \in \mathrm{Cr}(v_{G^0})$. The proof will be completed if we show that $x \in \mathrm{Cr}(v_G)$. Since $\beta_1(G^0) = \beta_1(G) < \frac{1}{2}n$, we have $n_1 < n_2$ where $n_1 = |N_1|$, $n_2 = |N_2|$. Two sets N_1 and N_2 which form the bipartition of N are noted in such a way that $n_1 \leq n_2$.

The inequality $n_1 < n_2$ implies that N_2 contains a point P_i which is game-excludable in G^0 so that $x_i = 0$. Hence, $x_i = 0$ for all $P_i \in N_2$ and $x_i = 1$ for all $P_i \in N_1$. As a consequence, $\beta_1(G) = \beta_1(G^0) = v_{G^0}(N) = x_1 + \cdots + x_n = n_1$. The identity $\beta_1(G) = n_1$ implies that all points in N_1 are game-nonexcludable in G. The next steps are same as those made at the end of the proof of Theorem 7.1, namely, it is demonstrated that all points in N_2 are game-excludable in G and hence they are not tied among one another. □

8. The taxonomy of game-indecomposable homogenous one-exchange networks

An examination of the situations in which the equation $Ax = w$ has no, many or exactly one solution has led Bonacich (1998, 1999) to distinguish four types of indecomposable networks: (i) "coreless" networks for which this equation has no solution; (ii) "strong power" networks defined by the condition: equation $Ax = w$ has a unique solution and G^0 is bipartite; (iii) "equal power" networks defined by the condition: equation $Ax = w$ has a unique solution and G^0 is not bipartite; (iv) "indeterminate power" networks characterized by infinitely many solutions.

Consider the one-exchange network with the transaction opportunity graph displayed in Fig. 1. B_1B_2 and B_3B_4 are the only suboptimal lines. The remaining lines lie between two sets $N_1 = \{A_1, B_3, B_4\}$ and $N_2 = \{A_2, B_1, B_2\}$. Since the equation $Ax = w$ has infinitely many solutions of the form $x_{A_1} = x_{B_3} = x_{B_4} = a$, $x_{A_2} = x_{B_1} = x_{B_2} = 1 - a$, Bonacich would classify this indecomposable network as indeterminate power. However, not all solutions are in the core, but only those for which $x_{B_1} + x_{B_2} \geq 1$ and $x_{B_3} + x_{B_4} \geq 1$, that is, $1 - a + 1 - a \geq 1$ and $a + a \geq 1$, Hence, $1 \geq 2a$ and $2a \geq 1$ so that $a = \frac{1}{2}$. Therefore, the core consists of the single payoff vector with all coordinates equal to $\frac{1}{2}$. The conclusion is that the network under consideration does not differ in this respect from type III networks in Bonacich's partition. For this reason, we propose to correct the taxonomy of game-indecomposable networks in such a way that structural types reflect directly the shape of the core.

The first of five classes that are defined below is formed by *coreless networks*, or those for which the core of the associated game is empty. According to Theorem 7.1, this is the case if and only if equation $Ax = w$ has no solution. Thus, coreless networks are exactly Bonacich's type I.

We define in turn *strong game–power networks* as such networks that the core of the associated game is formed by exactly one 0–1 payoff vector. The characterization of this class is given by the following theorem.

Theorem 8.1. *For every game-indecomposable homogenous one-exchange network G, the following conditions are equivalent*:

(1) *G is strong game–power*;
(2) $\beta_1(G) < \frac{1}{2}n$ *and* $\beta_1(G) = \alpha_0(G)$;
(3) G^0 *is bipartite and the sets in the partition of N are unequal size*.

The demonstration of Theorem 8.1 is left to the reader. Theorems 6.2 and 7.2 and the tools we used to prove them make this task fairly easy.

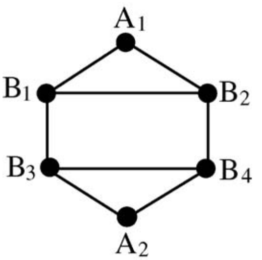

Fig. 1. Example of an equal game–power network.

Strong game–power networks coincide with Bonacich's type II networks. The term "strong *game*-power" is used here instead of Bonacich's original term "strong power" to mark the difference between the game-theoretic concept and the concept of "strong power" defined in the context of the theory which uses *maximal* matchings to define power in a one-exchange network.

Recall that *elementary power* was defined at the end of Section 1 – as a binary relation in the set of nodes of a one-exchange network – by the following condition: "P can exclude Q and Q cannot exclude P" where "P can exclude Q" means that there exists a maximal matching T which covers P and does not cover Q. By replacing "maximal" with "optimal" in the above statement we obtain the definitions of the *game-exclusion* and *game–power* relations. Like the elementary power relation the game–power relation is asymmetric and transitive.

Bonacich and Bienenstock did not explicitly define the game–power relation. In their formal theory of exchange networks, purely mathematical considerations are interwoven with empirical predictions, namely, the authors claim that in some one-exchange networks observed payoff distribution should be highly imbalanced due to some structural properties of the associated game. In particular, the members of the "upper class" in a strong game–power network who are all game-nonexcludable are predicted to have overwhelming payoff advantage over the members of the "lower class" who are all game-excludable.

To define the strong variety of elementary power, we need to define first the *exchange-seek relation*. We do this via the following statement: *P seeks exchange with all excludable neighbors* (all Qs such that $PQ \in L$ and Q is excludable) *or with all its neighbors if all are nonexcludable*. This condition formally renders the idea that a rational actor should be interested in trading with those neighbors whose bargaining power is as weak as possible. Since excludable neighbors are "weaker" than nonexcludable ones, the former will be preferred to the latter as transaction partners. If all P's potential partners are equally strong (that is, all Qs such that line PQ is in L are excludable or all are nonexcludable), P will negotiate with all of them, otherwise he will address his offers only to the weaker neighbors.

For example, in the 4-CHAIN network B_1–A_1–A_2–B_2, positions B_1 and B_2 seek exchange with A_1 and A_2, respectively, as their only available partners. A_1 whose potential partners are B_1 and A_2 seeks exchange with B_1 because B_1 is excludable while A_2 is nonexcludable. Similarly, A_2 seeks exchange with B_2. Note that A_i and B_i ($i = 1, 2$) seek exchange with each other, while A_1 and A_2 do not seek exchange with each other.

In general, the exchange-seek relation need not be symmetric. Consider another simple one-exchange network known as STEM in the network literature. It has four nodes structurally labeled A, B, C_1, C_2 and four lines AB, AC_1, AC_2, C_1C_2 so it can be drawn as a "triangle with a one-line tail." A, being the only nonexcludable position in this network, seeks exchange with its three excludable neighbors B, C_1, and C_2, but only B reciprocates A's choice, C_1 and C_2 seek exchange with each other.

4-CHAIN and STEM are two one-exchange networks which have been most frequently studied by mathematical sociologists (Markovsky et al., 1993; Skvoretz and Willer, 1991, 1993). They developed and refined their formal models of network exchange and tested them upon relevant data: records of experimental sessions run in a laboratory with the help of a special program allowing the subjects to communicate with one another via a computer network. Experimental evidence has confirmed the prediction that A_1 and A_2 in 4-CHAIN, and A and C_i in STEM should rarely trade with each other. Expected low frequency of exchanges in these lines can be derived from the theory which attributes this effect to bilateral disinterest or unilateral interest in a transaction (A_1 does not seek exchange with A_2 and conversely; A seeks exchange with C_i, C_i does not seek exchange with A). Notice that the same prediction can be deduced from the theory which claims that transactions should not happen in *suboptimal* lines. However, in some networks, the two theories give conflicting predictions.

Bienenstock and Bonacich did not formally define an exchange-seek relation as a basis for the formation of "trading patterns" as they call transaction sets. They claim that the network game is most likely to end up with the maximum number of transactions possible. In their experimental work, they assume incomplete information condition which implies that the subjects may not know which lines are optimal. An actor incompletely informed about the network shape cannot learn that his bargaining power rests on being game-nonexcludable or game-excludable, that is, being more or less indispensable to the group for maximizing the collective profit.

For any connected graph $G = (N, L)$, let L^m denote the subset of L made up of lines PQ such that P seeks exchange with *Q and Q* seeks exchange with P. G is said to be *power-indecomposable* if the subgraph $G^m = (N, L^m)$ of G is connected or *power-decomposable* if it is not so. In the latter case, the subgraphs of G generated by the node sets of connected components of G^m are called *power components* of G.

Let us examine two one-exchange networks whose transaction opportunity graphs are displayed in Fig. 2. Both networks are game-indecomposable and strong game–power. They are drawn so as to reveal their common two-class game–power structure. 5-CHAIN is power-indecomposable, 7-CHAIN has 3 power components generated by $\{C_1, D_1\}$, $\{C_2, D_2\}$, and $\{A, B_1, B_2\}$.

Consider again an arbitrary one-exchange network over $G = (N, L)$. We say that a matching T *is acceptable to* P if there is a line PQ in T such that P seeks exchange with Q and Q seeks exchange with P. The elementary power of P over its neighbor Q is called *strong* if (1) Q seeks exchange with P; (2) there exists a maximal matching T such that T covers P and does not cover Q, and T is acceptable to P and to all nonexcludable positions in the same power component.

Elementary power that is not strong is called *weak*. These definitions should put an end to the long-lasting search (Willer, 1999) of a proper criterion for the distinction between strong and weak variety of *exclusionary* power. The need for such a distinction appeared with the discovery that A's power over B_i in the 3-CHAIN (2-STAR) network B_1–A–B_2 produces a much greater payoff imbalance than the power of A_i over B_i in the 4-CHAIN network B_1–A_1–A_2–B_2. The latter network has two power components. The power of A_i over B_i is weak because the only maximal matching under which a B_i is excluded is not acceptable to A_i.

A power-indecomposable one-exchange network in which there appears only strong variety of elementary power is called *strong power*. The remaining structural types of power-indecomposable networks (and of power components in power-decomposable networks) are: *weak power*, *mixed power*, *equal power* (with all points nonexcludable), and *indeterminate power* (with all points excludable).

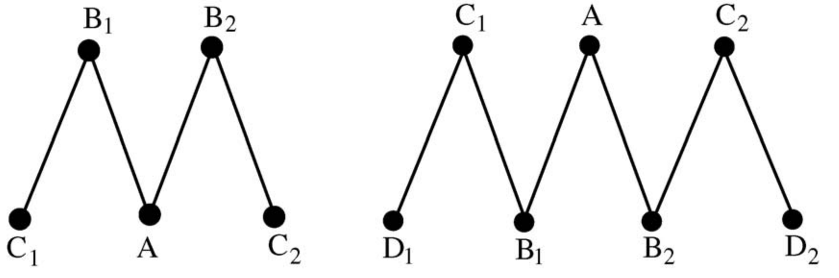

Fig. 2. Two strong game–power networks: 5-CHAIN and 7-CHAIN.

It is easy to verify that the 5-CHAIN network is strong power. One can prove that *every strong power network is strong game–power* so that 5-CHAIN is strong game–power as well. The case of the 7-CHAIN network which is strong game–power shows that this implication cannot be reversed.

The power of C_i over D_i is weak. A is excludable as it is not covered by the matching $\{B_1C_1, B_2C_2\}$ which is maximal but not optimal. Since all other maximal matchings are optimal and cover A, A is game-nonexcludable. Positions C_1 and C_2 are nonexcludable and hence game-nonexcludable. According to Bienenstock and Bonacich the occupants of all three game-nonexcludable positions A, C_1, and C_2 should exploit their partners in positions B_i and D_i. However, experimental data (Skvoretz, 2000) contradict this prediction: A's mean earning in transactions with the B_i slightly exceeded 12 points (half the 24-point pool), while the C_i earned on average some 14 points in transactions with the B_i and D_i. Thus, game-nonexcludability does poorly as determinant of payoff level. Marginal payoff advantage of A over the B_i observed by Skvoretz can be explained by the effect of "parametric power."

The term *structural parameter of a node* is referred to any real-valued map F of N such that $F(P) = F(Q)$ if $Q = \pi(P)$ for some automorphism π of G. F is called a *power parameter* if $F(P) > F(Q)$ for any P and Q such that P has elementary power over Q, and $F(P) = F(Q)$ for any P and Q such that P and Q are in the *elementary equipower relation*, which means inability to exclude each other (equivalently, P and Q are nonexcludable). Elementary power, the inverse relation, the elementary equipower relation, and the *mutual excludability relation* (P can exclude Q and Q can exclude P, or, equivalently, P and Q are excludable) are called four *elementary relations*. Every power parameter F generates a *parametric extension* of the elementary power relation (elementary equipower relation) which is defined by the condition $F(P) > F(Q)$ ($F(P) = F(Q)$). Given a probability distribution on the set of maximal matchings (Friedkin, 1992; Markovsky, 1992), one can define for any P a power parameter, called a *Probability Power Index* whose value for P is the sum of the probabilities of all maximal matchings which cover P.

Let us proceed to define other structural types of game-indecomposable networks. G is called *equal game–power* if the core of v_G consists of one payoff vector $x_i = \frac{1}{2}$, $i = 1, \ldots, n$. Equal game–power networks comprise Bonacich's type III networks, or the networks such that v_G has a nonempty core and G^0 is *not* bipartite. The complete 4-node graph K_4 is the smallest representative of this first subclass of equal game–power networks, whereas the "hexagon with two parallel diagonals" (Fig. 1) is the smallest member of the second subclass containing equal game–power networks with bipartite G^0. The characterization of all equal game–power networks is given by the following theorem.

Theorem 8.2. *A game-indecomposable one-exchange network G is equal game–power if and only if* $\beta_1(G) = \frac{1}{2}n$ *and* $\beta_1(G) < \alpha_0(G)$.

Proof. G is either coreless or strong game–power if and only if $\beta_1(G) < \frac{1}{2}n$. Assume, therefore, that $\beta_1(G) = \frac{1}{2}n$. We must prove that G is equal game–power if and only if $\beta_1(G) < \alpha_0(G)$. Assume first that G^0 is bipartite. Then one of three cases may occur.

1. Neither N_1 nor N_2 does contain a suboptimal line. Then the core consists of all payoff vectors such that $x_i = a$ for $P_i \in N_1$ and $x_i = 1 - a$ for $P_i \in N_2$, for all a in the range from 0 to 1.
2. N_1 or N_2, but only one of two sets contains a suboptimal line. If such a line is found within N_1, then the core consists of payoff vectors such that $x_i = a \geq \frac{1}{2}$ for all $P_i \in N_1$ and $xi = 1 - a$ for all $P_i \in N_2$.

3. Suboptimal lines occur within both N_1 and N_2, and the core reduces to the single payoff vector $x_i = \frac{1}{2}$.

Note that $\alpha_0(G) = \beta_1(G)$ in cases 1 and 2, while $\alpha_0(G) > \beta_1(G)$ in case 3. Therefore, Theorem 8.2 holds valid for bipartite G^0.

Assume in turn that G^0 is not bipartite. We already know that then G is equal game–power, so we do not need to make use of the condition $\alpha_0(G) > \beta_1(G)$. However, we should prove that this condition is necessary also in the case of non-bipartite G^0. Suppose for indirect proof that $\alpha_0(G) = \beta_1(G)$. Let T be an optimal matching in G, and let S be a minimum point cover. Since $\alpha_0(G) = \beta_1(G) = \frac{1}{2}n$, S consists of $\frac{1}{2}n$ points taken each from a line in T. There can be no line with both ends in $N - S$, for such a line would not be covered by S. To prove that G^0 is bipartite and thus obtain a contradiction, it suffices to show that all lines with both ends in S are suboptimal. The assumption $\beta_1(G) = \frac{1}{2}n$ means that any optimal matching covers all points. The points in $N - S$ which are not tied among one another can be covered only by the lines having each the other end in S. If a line PQ with P and Q in S were optimal, then $\frac{1}{2}n$ points in $N - S$ had to be matched with at most $\frac{1}{2}n - 2$ points in S, which is impossible. □

The proof of Theorem 8.2 has revealed the structure of one-exchange networks which are not yet classified. They satisfy the condition $\beta_1(G) = \frac{1}{2}n$ *and* $\beta_1(G) = \alpha_0(G)$ and form two classes we call *weak game–power* networks and *indeterminate game–power* networks. In both cases G^0 is bipartite and the set of players decomposes into two subsets having each $\frac{1}{2}n$ members. If the players in one subset get a points, those in the complementary subset get $1 - a$ points. In game-indeterminate power networks, there is no line joining two points in the same set, so the core conditions are met whatever number a from the [0, 1] interval is assigned to all members of one class of players.

DYAD and BOX are simplest indeterminate game–power game-indecomposable networks. DBOX is the simplest weak game–power game-indecomposable network. If G contains a line within one class, say, the one in which all players earn a units, then $a + a \geq 1$, which implies that $a \geq 1 - a$ so that the players who are given an opportunity to conclude a transaction not only with the members of the opposite group but also among themselves have a slight advantage over the latter. Thus, weak game–power networks have also a kind of two-class structure. The payoffs of the lower class members are at best equal to the payoffs of the upper class members. Equal game–power networks either do not have a class division or the two classes are equally strong. In both cases all players get the same payoff equal to half the profit pool. In indeterminate game–power networks, the outcome of the match between the two "teams" equal in number is unpredictable on structural basis: either side can win more points than the other.

Bienenstock and Bonacich's original theory does not make a distinction between weak and strong variety of game–power in the general context. The game–power relation, which has been defined here without any reference to the core, can occur both in strong game–power and in coreless networks, yet not in the remaining three network classes which contain only game-nonexcludable positions.

Bonacich (1999: p.214) claims that in coreless networks "There should not be stable power differences or exchange patterns." I would restrict this prediction to those coreless networks in which all nodes are game-excludable. In coreless networks which have both node types, game-nonexcludable players should have some payoff advantage over their game-excludable neighbors, even if the outcomes of the game may vary to a greater extent than in the networks with nonempty core.

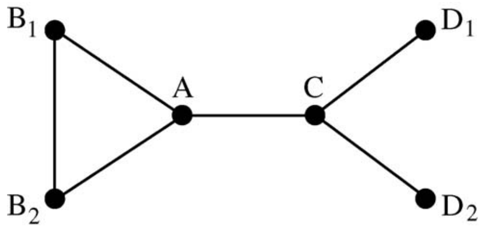

Fig. 3. A coreless network with embedded strong power subnetwork.

Consider, for example, the six-node coreless game-indecomposable network displayed in Fig. 3. It is natural to expect that the game-nonexcludable position C will have some advantage over D_1, D_2, and A. All three neighbors of C are game-excludable so that C should seek exchange with them as equally good members of the "lower class." D_1 and D_2 will seek exchange with C, their only potential partner, and the counterpart of the exchange-seek relation could be used to predict that a transaction between C and a D_i is more likely to occur than between C and A. The game-excludable B_1 and B_2 will be preferred by A to game-nonexcludable C, which will result in a network break at the AC line. Under the original theory which claims that only suboptimal lines will not be used for transactions, there is no reason to predict low frequency of exchanges between A and C because AC line is optimal like any other line in this network.

9. Decomposition of a one-exchange network into game-components

Consider now an arbitrary homogenous one-exchange network $G = (N, L)$. If $G^0 = (N, L^0)$ is not connected, then there exists a partition of N into pairwise disjoint nonempty subsets $N_1, \ldots, N_m$ such that all subgraphs $G_i^0 = G_{N_i}^0$ of G^0 generated by the N_i are connected. The subgraphs $G_i = G_{N_i}$ of G generated by the same subsets of N will be called *game-components* of G. A special term is needed (Bienenstock and Bonacich use the term "block" or "component") to distinguish them from *power components* defined in the previous section.

In general, a component of an unconnected graph may reduce to an isolated point. This is not possible for the components of G^0.

Theorem 9.1. *Each game-component of a homogenous one-exchange network has at least two points.*

Proof. Suppose that a single point P_i forms a game component. Since G is connected, P_i must be tied to a point P_j in another game component by a suboptimal line P_iP_j. P_j does not form another single point game-component for otherwise line P_iP_j could be added to any optimal matching in G, which is impossible. Thus, P_j lies within a game-component with at least two points and there exists an optimal matching T covering P_j. By replacing the line in T which covers P_j with P_iP_j we obtain a matching with the same number of lines as T. Hence line P_iP_j is optimal, which is a contradiction. □

Theorem 9.1 is important due to the corollaries one can derive from it. First, all hanging lines are optimal. PQ is a *hanging line* if $\deg(P) = 1$ and $\deg(Q) > 1$ or $\deg(P) > 1$ and $\deg(Q) = 1$, where $\deg(P) = |\{Q : PQ \in L\}|$ stands for the *degree* of node P.

The second corollary is that every P is covered by an optimal matching, in other words, every player can contribute to the maximization of group payoff.

In any superadditive game (N, v), the players can be divided into three categories with respect to how the maximization of the group payoff depends on each player. The least "powerful" players are those who are not members of any minimal coalition S such that $v(S) = v(N)$. The second set consists of those players who participate each in at least one minimal coalition but not in all minimal coalitions with this property. The third set is formed by the players who are members of every minimal coalition maximizing the group payoff. The game associated with a homogenous one-exchange network may have only the second and third type of players. This distinction is exactly that between game-excludable and game-nonexcludable positions.

To each game-component $G_i = (N_i, L_i)$ of a connected graph $G = (N, L)$ there corresponds the Bienenstock–Bonacich game (N_i, v_{G_i}). It is not difficult to verify that v_{G_i} coincides with the restriction of v_G to the subsets of N_i.

If T is an optimal matching in G, then the intersection of T and L_i is an optimal matching in G_i. Hence, each optimal line in G which lies in L_i is optimal in G_i. On the other hand, the union of optimal matchings taken from all game-components is an optimal matching in G. Therefore, a line in L_i is optimal in G_i if and only if it is optimal in G. The type of a position in a game-component coincides with its type in the whole network. For any point P in G_i, P is game-(non)excludable in G_i if and only if P is game-(non)excludable in G. The theorems which follow are simple consequences of these facts.

Theorem 9.2. *The game-components of a homogenous one-exchange network are game-indecomposable.*

Theorem 9.3. *If $G_1, \ldots, G_m$ are game-components of G, then $v_G(N) = \sum_i v_{G_i}(N_i)$, or, in graph-theoretic terms, $\beta_1(G) = \sum_i \beta_1(G_i)$.*

In view of Theorem 9.2 the taxonomy of game-indecomposable networks applies to game-components of G as well. The type of a game-component G_i can be recognized by the values of two structural parameters β_1 and α_0 except for types 4 and 5 which differ with the location of suboptimal lines. Thus, every game-component G_i of G can be one of following types:

1. Coreless: $\beta_1(G_i) < \frac{1}{2}n_i$ and $\beta_1(G_i) < \alpha_0(G_i)$;
2. Strong game–power: $\beta_1(G_i) < \frac{1}{2}n_i$ and $\beta_1(G_i) = \alpha_0(G_i)$;
3. Equal game–power: $\beta_1(G_i) = \frac{1}{2}n_i$ and $\beta_1(G_i) < \alpha_0(G_i)$;
4. Weak game–power: $\beta_1(G_i) = \frac{1}{2}n_i$ and $\beta_1(G_i) = \alpha_0(G_i)$;
5. Indeterminate game–power: $\beta_1(G_i) = \frac{1}{2}n_i$ and $\beta_1(G_i) = \alpha_0(G_i)$.

The global game–power structure of G depends on the types of G's game-components and on the location of lines connecting them. However, according to Theorem 9.4, these suboptimal lines can connect only game-nonexcludable nodes in distinct game-components.

Theorem 9.4. *A game-excludable point in G can be tied only to points in the same game-component of G.*

Proof. Consider two points P and Q in two distinct game-components G_i and G_j. If both P and Q are game-excludable, then P is game-excludable in G_i and Q is game-excludable in G_j. This implies the existence of an optimal matching T in G covering neither P nor Q. If line PQ

were in G, it could be added to T, and T would not be optimal. If P is game-excludable and Q is game-nonexcludable, we consider an optimal matching T which covers Q and does not cover P. If line PQ were in G, then the line in T covering Q in G_j could be replaced with PQ to obtain from T a matching T' containing PQ and having as many lines as T. Then line PQ would be optimal, but all lines joining points in distinct components are suboptimal. □

Theorem 9.4 implies, in particular, that the dyads in game–power relation can be found only inside game-components (coreless or strong game–power).

We are now in a position to prove the main theorem which together with Theorem 6.2 provides a complete graph-theoretic characterization of one-exchange networks for which the Bienenstock–Bonacich game has a nonempty core.

Theorem 9.5. *The game (N, v_G) representing a homogenous one-exchange network over $G = (N, L)$ has a nonempty core if and only all games (N_i, v_{G_i}) representing G's game-components $G_i = (N_i, L_i)$ have nonempty cores.*

Proof of necessity. Assume that the payoff vector x is in the core of (N, v_G). Since Theorem 9.3 implies that $v_G(N) = \sum_i v_{G_i}(N_i)$, x is feasible for the coalition structure $\{N_1, \ldots, N_m\}$. As a consequence, the sum of x_j over all P_j in N_i equals $v_G(N_i) = v_{G_i}(N_i)$. Therefore, the x payoffs which go to the players in N_i satisfy the (group rationality) condition 3 in Theorem 5.3. Clearly, condition 1 (individual rationality) and condition 2 (dyadic rationality) are also met and the proof of necessity is completed.

Proof of sufficiency. Assume that every game (N_i, v_{G_i}) has a nonempty core. We construct the payoff vector for (N, v_G) by combining the payoffs vectors taken each from the cores of particular game-components. Since the core payoffs allotted to the players in each N_i sum up to $v_{G_i}(N_i)$, the overall payoff sum equals $v_G(N)$ by Theorem 9.3. Therefore, any payoff vector constructed in such a way satisfies the group rationality condition. It remains to be shown that the core payoffs in each game-component can be selected so that the dyadic rationality condition be met for any two players from distinct game-components. Recall now that the payoff vector with all coordinates equal to $\frac{1}{2}$ is always in the core of a noncoreless game-component except for strong game–power components where 0 and 1 are the only possible core payoffs. If the $\frac{1}{2}$ solution is chosen for equal, weak and indeterminate game–power components, the problem with satisfying the inequality $x_P + x_Q \geq 1$ by a line PQ with P and Q in distinct game-components might arise only if $x_P = 0$ or $x_Q = 0$, that is, if P or Q were a game-excludable node in a strong game–power component. However, such lines do not exist in virtue of Theorem 9.4.

For odd n, coreless networks occur fairly frequently in the set of n-node connected graphs. To determine the numbers given in Table 1, I wrote a computer program for which I used as input the list of adjacency matrices (generated by John Skvoretz, see Skvoretz, 1996) of all nonisomorphic connected graphs with the number of nodes ranging from 2 to 8.

According to Theorem 9.5, to check if a one-exchange network is coreless, one has to decompose it into game-components and verify if each component satisfies the necessary and sufficient condition given in Theorems 6.2, 7.1 or 7.2. Only 15 out 921 coreless networks with up to 8 nodes have 2 game-components, all other are game-indecomposable of which 797 have no suboptimal lines.

If all game-components have nonempty cores, the core for the whole network can be constructed by imposing additional constraints on payoffs in those game-components for which the core does

Table 1
The number of structurally distinct small connected graphs and the respective number of coreless one-exchange networks

	n							Total
	2	3	4	5	6	7	8	
Connected graphs	1	2	6	21	112	853	11117	12112
Coreless networks	0	1	0	12	3	626	279	921

not reduce to one payoff vector. We illustrate this, using the 4-CHAIN network, the simplest one-exchange network with two game-components (A–B_1 and A_2–B_2), both of indeterminate type.

Let a_i and $1 - a_i$ ($i = 1, 2; 0 \leq a_i \leq 1$) stand for the payoffs of A_i and B_i in the components' cores. A payoff vector $(1 - a_1, a_1, a_2, 1 - a_2)$ is in the core of the 4-CHAIN game if and only if a_1 and a_2 satisfy the inequality $a_1 + a_2 \geq 1$. The combination of the two cores can be represented geometrically by the unit square $[0, 1] \times [0, 1] = \{(a_1, a_2) : 0 \leq a_1 \leq 1, 0 \leq a_2 \leq 1\}$. Under the constraint $a_1 + a_2 \geq 1$, which establishes a kind of dependence between the payoffs in two game-components, the range of admissible outcomes shrinks to the triangle $\{(a_1, a_2) : 0 \leq a_1 \leq 1, 1 - a_1 \leq a_2 \leq 1\}$ with vertices $(1, 0)$, $(0, 1)$, $(1, 1)$.

In two experiments on 4-CHAIN (Skvoretz and Willer, 1991, 1993), in eight sessions with eight distinct groups, maximal matchings occurred in 114 out of 126 rounds. Only in 10 out of 89 rounds in which optimal matchings were observed, the total payoff of A_1 and A_2 did not reach the level of 24 points (the pool size commonly used in experiments). Thus, the number of games which ended up with an outcome in the core is high enough to support Bienenstock and Bonacich's *empirical* exchange theory. The outcomes (a_1, a_2) such that both A_1 and A_2 gain no less than their partners B_1 and B_2 lie in the square with vertices $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(1, 1)$, $(1, \frac{1}{2})$. Its area $(\frac{1}{4})$ is exactly half the area of the triangle representing the core of the 4-CHAIN game. If we assume that all payoff vectors in the core are equally likely and use the geometric probability distribution to formalize this assumption, then the probability of A_1 and A_2's *simultaneous* payoff advantage over B_1 and B_2 will be $\frac{1}{2}$. Since this event happened in 66 (83.5%) out of 79 rounds, the core theory appears a rather imperfect tool for predicting negotiation results for dyads in experimental one-exchange networks.

10. Concluding remarks, one more theorem, and further issues

While economic theory has always focused on free markets, or exchange systems in which every two actors with complementary interests are allowed to trade with each other, sociologists have been more interested in studying exchange processes in social systems in which partner choice and bargaining are subject to some structural constraints suspected to generate uneven distribution of rewards. Mathematical modeling of exchange systems endowed with *network* structure requires a different language than that used by economists for modeling market equilibrium. According to Richard Emerson, the pioneer of network approach to social exchange,"The study of exchange networks might be formalized through a set of coordinating definitions joining networks to graph theory, thereby gaining access to the mathematical properties of that formal theory." (Emerson, 1972: p. 71). His own preliminary strategy consisted in the use of "digraphs largely as representational devices, with occasional utilization of concepts from graph theory as a prelude to more thorough formalization." (Emerson, 1972: p. 71). Subsequent attempts to find a "more thorough" formalization paved the way for the most sophisticated use of mathematics:

the representation of *empirical* objects of a given type by *mathematical* objects of a well defined *category*, constructed so as to formally model special properties of empirical systems under study.

My aim in this paper was to work out a formalism and a mathematical *theory* for just one of *three major strands* in network exchange theorizing. The first of them, which can only be mentioned here, has grown out of Emerson's concept of dependence. Mathematical elaboration of these ideas (Cook and Yamagishi, 1992; Sozański, 1997) leads to the theory of abstract *dynamical systems* of the form (X, T) where X – the system's *space of states* – is a set endowed with a *topological* structure, and T is a *continuous mapping* of X into X. For more see the preprint of Chapter 4 of the author's book.

The second path in mathematical theorizing on exchange networks opened with the appearance of Markovsky, Willer and Patton's seminal paper (1988) and culminated in the publication of the book *Network Exchange Theory* (1999). This volume, edited by David Willer, contains old and new papers of a group of sociologists (Lovaglia, Markovsky, Skvoretz, Willer) working under the banner of Elementary Theory (Willer and Markovsky, 1993). They proposed a refined quantification of "exclusionary power" but failed to ultimately clarify the distinction between its "strong" and "weak" variety. New algorithms they proposed (*Iterative GPI Analysis* and related procedures) to solve this perplexing problem show that even the highest level of methodological awareness does not protect sociologists against the temptation to complicate formal constructs beyond reasonable measure.

A couple of new simple definitions and solutions pertaining to this problem area were presented in this article (theorems and proofs can be found in the preprint of Chapter 3 of my monograph), yet my main objective was to develop the third mathematical approach, proposed by Bienenstock and Bonacich, which rests on *representing* an exchange network by an n-person game in characteristic function form.

Most theorems proven in this paper characterize the core of the games associated with *homogenous one-exchange networks*, the class of exchange networks on which sociologists have done most experiments. Unlike other solutions, which need not be "network-feasible," the core, provided that it is not empty, can be directly applied to predict payoffs and trading patterns in empirical network games. Unfortunately, the core theory, confronted with experimental evidence, loses the contest with theories, formulated within the second brand of network exchange theorizing, namely, the theories which (1) relate an actor's profit to the amount of bargaining power of his network position (exclusionary power is quantified by means of the "likelihood of being included in an exchange") and (2) derive the predicted pool split in each dyad from the principle of balancing resistance (Lovaglia et al., 1995).

What is then the use of the *theory of generalized assignment games* for the *sociology* of exchange networks? It is not my intention to deny *any* empirical validity to the theory which makes use of the core to predict some properties of the exchange process. Indeed, Bonacich and Bienenstock (1995) provided strong experimental evidence for their claim that the dynamics of network exchange much depends on whether the network game does or does not have a nonempty core. In coreless networks, negotiation outcomes are more "unstable," which means that trading patterns and payoffs vary to a greater degree between rounds when the game is played many times by the same players.

The concept of stability has a well defined meaning in the theory of characteristic function games. A set Z of payoff vectors is *internally stable* if no element in Z is dominated by any other element in Z. A payoff vector x is said to *dominate* a payoff vector y if there exists a coalition S such that $x_i > y_i$ for all $P_i \in S$, and x is feasible for S, that is, $\sum(S, x) \leq v(S)$. The core is often defined equivalently as the set of undominated imputations. Indeed, one can easily prove (see Owen, 1995: Chapter X) that *a payoff vector x is in the core if and only if there is no payoff vector dominating x*.

Inspired by Bonacich's paper (1998), I define in turn a stable outcome of a negotiation round in a one-exchange network. Let x be a payoff vector feasible for a matching T. Let us call the pair (x, T) *stable* if there do not exist a matching U and a payoff vector y feasible for U such that $y_P > x_P$ and $y_Q > x_Q$ for some line PQ in U. The definition of stability (note that it is applicable also outside the context of the characteristic function representation of an exchange network) implies that an outcome (x, T) of a negotiation round is unstable if and only if at least two actors, by making a deal between themselves, could *both* gain more points than they managed to negotiate from their partners with whom they are matched under T.

If x is in the core and T is a network-optimal matching, then (x, T) is stable, which is a simple consequence of the theorem equating the core with the set of undominated payoff vectors. Is the inverse implication also true? Let x be a payoff vector and let T be a matching such that $x_i + x_j = C_{ij}$ for any line P_iP_j in T and $x_h = 0$ for any point P_h not covered by T. Assume that (x, T) is stable and suppose that x is not in the core. Then, according to Theorem 5.3, there exists a network line P_kP_l such that $x_k + x_l < C_{kl}$. Putting $y_k = x_k + e$, $y_l = x_l + e$, where $e = \frac{1}{2}(C_{kl} - x_k - x_l)$, $y_h = 0$ for $h \neq k, l$, we get a payoff vector y feasible for $U = \{P_kP_l\}$ such that $y_k > x_k$ and $y_l > x_l$, which means that (x, T) is not stable. Thus, we have proven the following theorem.

Theorem 10.1. *For any one-exchange network, if x is a payoff vector feasible for a matching T, then (x, T) is stable if and only if x is in the core of the associated game.*

Bonacich (1999: pp. 208–210) claims that his behavioral theory of network exchange is grounded solely on the principles of *individual* rational decision making. Theorem 10.1 can be invoked to justify his claim in relation to network games with nonempty cores. To predict that such a game will end up with a network-optimal matching, one can do without attributing to the players a kind of collective orientation, that is, there is no need to assume that the bargainers, interested in increasing their individual payoffs, are also concerned about maximizing the total group profit.

Markovsky (1997: p. 67), in reply to Bienenstock and Bonacich (1997) blamed them of making "explicit attempts to supplant network exchange formulations." Seen in the *mathematical* perspective, the two paths of formal network exchange theorizing show more convergence than it appears to the proponents of competing *behavioral* theories. The generalized assignment *game*, defined as a *representation* of a one-exchange *network*, is an auxiliary tool rather than alternative model.

The main theorems proven in this paper show that the study of the games associated with one-exchange networks leads back to graph theory, specifically, to its classical chapters on *coverings* and *matchings* (Berge, 1973: Chapter 7; Harary, 1969: Chapter 10; Ore, 1963: Chapter 4). Busacker and Saaty (1965: p.170) wrote 40 years ago: "So far, matching is a theoretical subject. It is an excellent example of a mathematical theory that needs an application." Today we know that this old theme has found applications in *empirical* network exchange theories. Moreover, the need to formalize the sociological concept of power inspired further work in the relevant area of graph theory, just as the discovery of "peck order" gave a spur to the development of the "theory of tournaments."

Formalized empirical network exchange theories should be counted among most important achievements of *mathematical sociology*. Unfortunately, many of these theories still remain little known to general sociological audience. The reader of Jonathan Turner's account of social exchange theory (Turner, 1998: Part IV) will find there only a footnote which marks the mere existence of those theories that do not fit into the Emersonian tradition.

Bienenstock and Bonacich's *empirical core theory* can be placed within a broad class of network exchange theories which (1) build on the characteristic function representation of a one-exchange network; (2) presume or imply group rationality condition; (3) restrict the set of outcomes to payoff vectors feasible for matchings, and, as a consequence of these conditions, predict that the group interaction process in a homogenous network should end up with an *optimal* matching. It is natural to postulate for such theories that the positions which are covered by *all* optimal matchings have some "power advantage" over the positions that are not indispensable for maximizing the number of transactions.

By contrast, the theory of exclusionary power builds on the distinction between nonexcludable positions (those covered by all *maximal* matchings) and excludable positions. As shown in Section 9, some networks contain excludable positions which are classified Bienenstock and Bonacich's core theory as advantaged along with nonexcludable positions. Thus, the choice of the core for predicting payoff imbalance may be inappropriate insofar as one would like to bring together the theories which use the game representation of a one-exchange network and the theories which locate the source of power in unequal vulnerability to exclusion.

Bienenstock and Bonacich (1993) considered also other solutions for generalized assignment games. The *Shapley value*, the best known solution besides the core, unlike the latter, exists for every characteristic function game as a single payoff vector uniquely determined by few natural postulates (see Owen, 1995: XII.1). Given a homogenous one-exchange network over a graph G, one can assign to any point P of G its Shapley payoff s_P, obtaining in such a way a new graph-theoretic structural parameter (recall that "structural" means that $s_P = s_{\alpha(P)}$ for any automorphism α of G).

The Shapley solution for a generalized assignment game differs from the core in that it need not be network-feasible. Therefore, s_P cannot be identified with P's profit from a transaction with one of its network neighbors. What can be done in these circumstances? I suggest: (1) to interpret s_P as the minimum payoff expected by an actor in position P, and (2) to determine P's theoretical payoff in a transaction with Q from the formula $x_{PQ} = \frac{1}{2}(C + s_P - s_Q)$. This formula was derived by Lovaglia and associates (1995) from the *equation of equal resistance* which is the second building block of their theory, the first being the definition of the *minimum profit expected by P* as $\frac{1}{2}C$ multiplied by ESL(P), or P's "exchange-seek likelihood of being included in an exchange."

Does the Shapley value, often regarded as a solution reflecting equity considerations, have anything to do with *exclusionary* power? For now I can state the following conjecture which I found to be true for all connected graphs with up to 8 nodes: *If P has elementary power over Q, P seeks exchange with Q, and Q seeks exchange with P, then* $s_P > s_Q$.

Can we expect from the Shapley value to be a better predictor of reward distribution in experimental networks than ESL or any other probability power index? The Shapley value discriminates between nonexcludable positions for which any PPI takes the same value of 1. As a consequence, the theory which relates the division of a profit pool in a power dyad PQ to *two* variables s_P and s_Q may do better in predicting the size of payoff differential than the theory which resorts to *one* variable PPI(Q) (PPI(P) = 1 if P has elementary power over Q and PQ is a line of G). An examination of available experimental results generally confirms this expectation and justifies Bienenstock and Bonacich's (1997: p. 61)conclusion that "game theory has much to contribute to the study of exchange networks." To put it in more generally, an appropriate blending of game-theory *behavioral* models and graph-theory *structural* models can yield more refined theories able to explain the operation of social interaction systems with network structure.

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